ELEMENTARY

CO-ORDINATE GEOMETRY,

FOR

COLLEGIATE USE AND PRIVATE STUDY.

BY

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MAXIMUM REASONING, MINIMUM RECKONING.

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PREFACE.

IN the study of Analytic Geometry, as of almost anything else, either or both of two ends may be had in view: gain of knowledge, culture of mind. While the first is in itself worthy enough, and for mathematical devotees all sufficient, it is certainly of only secondary importance to the mass of college students. For these the subject can be wisely prescribed in a curriculum only in case the mental drill it affords be very high in order of excellence.

The worth of mere calculation as an exercise of reason can hardly be considerable, for reason is exercised only in a tread-mill fashion. Even the solution of problems by algebraic processes is a very inferior discipline of reason, for only in forming the analytic statement does the reasoning rise clearly into consciousness; the operations that follow conduct one to the conclusion, but — with his eyes shut. In this respect Geometry is certainly a better discipline than Algebra, and the Euclidean than the Cartesian Geometry. But not in any kind of reasoning is the very best discipline found. No argument presents difficulty or calls for much mental effort to follow it, when once its terms are clearly understood; for no such argument can be harder to understand than the general syllogism of which it is a special case, and that is of well-known simplicity. The real difficulty lies in forming clear notions of things; in doing this all the higher faculties are brought into play. It is this formation of concepts, too, that is the really important part of mental training. He who forms them clearly and accurately may be safely trusted to put them together correctly.

Logical blunders are comparatively. rare. Nearly every seeming mistake in reasoning is really a mistake in conception. If this be false, that will be invalid.

It is considerations like the above that have guided the composition of this book. Concepts have been introduced in abundance, and the proofs made to hinge directly upon them. Treated in this way, the subject seems adapted as hardly any other to develop the power of thought.

The correlation of algebraic and geometric facts has been kept clearly and steadily in view. While each may be taken as pictures of the other, the former have generally been treated as originals, lending themselves much more readily to classification.

Only natural logical order has been aimed at in the development of the subject; no attempt has been made to keep up the distinctions of ancient and modern, analytic and synthetic.

With every step forward in Geometry the difficulty and tedium of graphical representation increases, while more and more the reasoning turns upon the *form* of the algebraic expressions. Accordingly, pains have been taken to make the notation throughout consistent and suggestive, and Determinants have been used freely.

By all this effort to make the book an instrument of culture, its worth as a repertory of mathematical facts has scarcely suffered; in this regard, as in others, comparison with other texts is invited.

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CHAPTER II.

INTRODUCTION.

DETERMINANTS.

Permutations.

1. Two things, as a and b, may be arranged straight, in the order of before and after, in but *two* ways: $a\,b$, $b\,a$. A *third* thing, as c, may be introduced into each of these arrangements in *three* ways: just before each or after all. Like may be said of any arrangement of n things: an $(n+1)$th thing may be introduced in $\overline{n+1}$ ways, namely, before each or after all. Hence the number of arrangements of $\overline{n+1}$ things is $\overline{n+1}$ times the number of arrangements of n things. Or

$$P_{n+1} = P_n \cdot \overline{n+1}.$$

Writing $n\,!$ for the product of the natural numbers up to n, we have

$$\overline{n+1}\,! = n\,!\,\overline{n+1}.$$

Hence, if $\quad P_n = n\,!, \quad P_{n+1} = \overline{n+1}\,!,\quad$ and so on.

Now $\quad P_2 = 2 = 1 \cdot 2 = 2\,!;$

hence, $\quad P_3 = 1 \cdot 2 \cdot 3 = 3\,!, \quad$ and $\quad P_n = n\,!$

The various arrangements of things in the order of before and after are called *straight Permutations* or simply **Permutations** of the things. The number of permutations of n things is $n\,!$ (read *factorial* n or n *factorial*).

If the things be arranged not straight but around, in a ring, we may suppose them strung on a string; if there are n of them, there are also n spaces between them. We may suppose the string cut at any one of the n spaces and then stretched straight; this will turn the *circular* permutation into a *straight* one; and since we may make n different cuts, each yielding a distinct straight permutation, the number of *straight* permutations of n things is n times the number of *circular* ones. Or

$$P_n = C_n \cdot n\,;\ \therefore C_n = \overline{n-1}\,!$$

2. The things, whatever they be, are most conveniently marked or named by letters or numbers. Of letters the alphabetic order is the *natural* order; of numbers the order of size is the *natural* order; as: $a, b, c, \cdots z$; $1, 2, 3, 4, \cdots n$.

If any change be made in either of these orders, say in the last, then some less number must appear after some greater, some greater before some less. Every such change from the natural order is called an **Inversion**. The number of inversions in any permutation is found by counting the number of numbers less than a number and placed after it, and taking the sum of the numbers so counted.

A permutation is named *even* or *odd*, according as the number of inversions in it is even or odd. Thus 2 5 3 1 6 4 is an even permutation containing 6 inversions; 3 1 2 5 4 6 is an odd permutation containing 3 inversions. The *natural* order, $1, 2, 3, \cdots n$, contains 0 inversions and is even; the *counter* order, $n \cdots 3, 2, 1$ contains $1+2+3+\cdots+n-1$ or $\dfrac{n \cdot \overline{n-1}}{1 \cdot 2}$ inversions and is *even* when the remainder on division of n by 4 is 0 or 1, *odd* when the remainder is 2 or 3.

It is plain that in any permutation any thing, symbol, or element may be brought to any place or next to any other one by exchanging it in turn with each of the ones between it and that other one. Thus, in 3 7 4 5 1 6 2, 7 may be brought next to 6 by exchanging it in turn with 4, 5, 1. Hence any permutation may be produced from any other by *exchanges of adjacents*.

3. By an exchange of any two *adjacents*, as p, q, the relations of each to all the others, and the relations of all the others among themselves, are not changed; only the relation of those two is changed. Now if pq be an inversion, qp is *not;* and if pq be *not* an inversion, qp is one; hence in either case, by this exchange of two adjacents, the number of inversions is *changed by* 1; hence the *permutation changes* from even to odd or from odd to even.

If p and q be *non*-adjacent, and there be k elements between them, then p is brought next to q by k exchanges in turn with adjacents, and then q is brought to p's former place by $\overline{k+1}$ exchanges with adjacents: p is carried over k elements and q over $\overline{k+1}$; thus p and q are made to exchange places by $2k+1$ exchanges of adjacents. The permutation meanwhile changes, from even to odd or from odd to even, $2\overline{k+1}$ times; and an *odd* number of changes back and forth leaves it *changed*. Hence, an *exchange of any two elements* in a permutation **changes** the permutation from even to odd or from odd to even.

Plainly all the permutations may be parted into pairs, the members of each pair being alike except as to p and q, which are exchanged in each pair; hence, *one* permutation of each *pair* will be *even*, and *one odd;* hence, of all the permutations, *half* are even, *half* odd.

Determinants.

4. It is plain that n^2 things may be parted into n classes of n each. We may mark these classes by letters: $a, b, c, \cdots n$, where it is understood that n is the nth letter; the rank of each in its class may be denoted by a subscript; thus p_k will be the kth member of class p. Clearly all members of rank k will also form a class of n members. The whole number may be thought arranged in a square of n rows and n columns, as in the special case $n = 5$, thus:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \\ a_5 & b_5 & c_5 & d_5 & e_5 \end{vmatrix}$$

This arrangement is not at all *necessary* to our reasoning, but is quite *convenient*.

Suppose we pick out of these n^2 things n of them, taking *one* of each *class* and *one* of each *rank* (clearly, then, we take *only one* of *each*). This we may do in $n!$ ways; for we may write off the n letters in natural order, $a, b, c, \cdots n$, and then suffix the subscripts in as many ways as we can permute them, i.e., in $n!$ ways.

Now suppose these n^2 things symbolized by letters to be magnitudes or numbers, and form the continued product of each set of n picked out as above: write off the sum of these products, giving each the sign $+$ or $-$ according as the permutation of the subscripts be even or odd: the result is called the **Determinant** of the magnitudes so classified. Accordingly a Determinant may be defined as

A sum of products *of n^2 symbols assorted into n classes of n ranks each, formed of factors taken one from each class and each rank, each product marked + or − according as the order of ranks* (or classes) *is an* **even** *or an* **odd** *permutation, the order of classes* (or ranks) *being natural.*

The symbols are called *elements* of the Determinant; each product, a *term*; the number of the *degree* of the Determinant is the number of *factors* in each product. The classes may be denoted by letters and the ranks by subscripts, or *vice versa*. The definition shows that *classes* and *ranks* stand on exactly like footing; in any reasoning they may be *exchanged*.

5. There are several ways of writing Determinants. In the square way, exemplified in Art. 4, the classes are written in *columns* and the ranks in *rows*, or *vice versa*. Hence *rows* and *columns* are always interchangeable. This is a very vivid way of writing them, but is tedious. It is shorter to write simply the diagonal term, thus:

$$\Sigma \pm a_1 b_2 c_3 \cdots n_n$$

The sign of summation Σ refers to the different terms got by *permuting the subscripts*, — there are $n!$ of them; the double sign $\pm$ means that each term is to be taken + or − according as the permutation of the subscripts is *even* or *odd*.

Still another way is to write the diagonal between bars:

$$|a_1 b_2 c_3 \cdots n_n|.$$

This is very convenient when there can be no doubt as to what are the elements not written: otherwise, the square form is best.

6. To exchange two rows in the square form would clearly be the same as to exchange in every term the indices or subscripts that mark those rows; but by Art. 3 this would *change each* permutation of the subscripts from even to odd or from odd to even; and this, by the definition, would *change the sign of each* term, and hence of the whole Determinant. Moreover, since rows and columns stand on like footing, the same holds of exchanging two columns: hence,

To exchange two columns (or rows) *changes the sign of the Determinant.*

If the two rows (or columns) exchanged be identical or congruent, i.e., if the elements corresponding in position in the two be equal each to each, clearly exchanging them can have no effect on the value of the Determinant, although it changes the sign; now the *only* number whose value is not changed by changing its sign is 0: hence,

The value of a Determinant with two congruent rows (or columns) *is* 0.

Every term of a Determinant contains one and only one element out of each row and column; hence a common factor in every element of a row (or column) must appear as a factor of every term of the Determinant and hence of the Determinant itself; hence, we may divide each element of the row (or column) by it, if at the same time we multiply the whole Determinant by it; i.e., *any factor of every element of a row* (or column) *of a Determinant may be set out aside as a factor of the whole Determinant.*

It is equally plain that any factor may be *introduced* into each element of a row (or column), if at the same time the whole Determinant be *divided* by that factor.

7. If we will find *all the terms that contain any one element* of the Determinant, say a_1, we may suppose all the other elements in its column and row to be 0; this will make vanish *no* term containing a_1 and *all* terms *not* containing a_1. The Determinant, say of 5th degree, will then be

$$\begin{vmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & b_2 & c_2 & d_2 & e_2 \\ 0 & b_3 & c_3 & d_3 & e_3 \\ 0 & b_4 & c_4 & d_4 & e_4 \\ 0 & b_5 & c_5 & d_5 & e_5 \end{vmatrix}$$

Setting aside a_1 as the first factor in each product, we find all the part-products by holding the order $bcde$ fast and permuting the subscripts $_{2\,3\,4\,5}$; but this is the way we form the Determinant $|b_2c_3d_4e_5|$; also the sign of each whole product, after multiplying by a_1, will be the same as the sign of the corresponding part-product, since 1 being in its natural place, the only possible inversion will be in the subscripts $_{2\,3\,4\,5}$. Hence, the sum of the part-products or multipliers of a_1 is the Determinant $|b_2c_3d_4e_5|$. It is called the **co-factor** (or sub-determinant, or minor) of a_1 and is the *Determinant left* after destroying the *row* and *column* of a_1.

If now we will find the co-factor of p_k, i.e., of the element in the pth column and kth row, we may bring its row to the first place by exchanging its row in turn with each row before it, and its column in turn with each column before it. By these $p+k-2$ exchanges the positions of the other rows and columns as to each other are not changed at all; they stand exactly as if the pth column and kth row had been destroyed. The new Determinant got by these exchanges will be the old one *term for term* with like or unlike sign according as $p+k-2$ (the number of changes of sign) is even or odd, or, what amounts to the same, *according as* $p+k$ is *even* or *odd;* hence the co-factor of p_k in the new Determinant will be the co-factor of p_k in the old with *like* or *unlike* sign according as $p+k$ is *even* or *odd*. But p_k is in the first column and first row of the new Determinant, hence, by the foregoing, its co-factor is got by destroying its column and row, i.e., by destroying the pth column and kth row of the old Determinant; hence the co-factor of p_k in the old Determinant is the *Determinant left* on destroying the *row* and *column* of p_k, but taken $+$ or $-$ according as $p+k$ is *even* or *odd.*

The co-factor of any element may be denoted by the same symbol written large; thus, the co-factor of p_k is P_k.

8. By definition, all the terms containing any element are got by multiplying that element by its co-factor; if, then, we multiply *each* element of a row (or column) by its co-factor and form the *sum*, we shall get *all the terms* of the Determinant that contain *any* element of that row (or column); but *every term* contains *one* element of that row (or column); hence we get *all terms of the Determinant;* and since *no term* contains *two* elements of that row (or column), we get *each* term but *once*. Hence, *the sum of products of each element of a row* (or column) *by its own co-factor* is the *Determinant* itself;

$$|a_1 b_2 c_3 \cdots n_n| = a_1 A_1 + b_1 B_1 + c_1 C_1 + \cdots + n_1 N_1, = \text{etc.}$$

It is to note that the co-factors subscribed $_1$ contain every other subscript in their values *but* 1, and every other letter *but* their own. Now change the subscript $_1$ to $_2$ on both sides of the equation; there results

$$|a_2 b_2 c_3 \cdots n_n| = a_2 A_1 + b_2 B_1 + c_2 C_1 + \cdots + n_2 N_1.$$

The subscripts of the co-factors are *not* changed, because the subscript $_1$ does *not* appear in their values.

Now the left side of this equation is a Determinant with two rows congruent, namely, the 1st and 2d, since the subscripts of the 1st, which were all $_1$ were changed to $_2$, the subscripts of the 2d; hence its value is 0 by Art. 6; i.e.,

$$a_2A_1 + b_2B_1 + c_2C_1 + \cdots + n_2N_1 = 0.$$

The small letters are the elements of the 2d row, the large letters are the co-factors of the corresponding elements of the 1st row; plainly the reasoning about columns or about any other pair of subscripts would be the same; hence *the sum of products of each element of a row* (or column) *by the co-factor of the corresponding element of any other row* (or column) is 0.

9. If the elements of any row (or column), as the 1st, be regarded each as the *sum* of two *part*-elements, so that

$$a_1 = a_1' + a_1'', \quad b_1 = b_1' + b_1'', \quad \cdots, \quad n_1 = n_1' + n_1'',$$

then we shall have

$$|a_1b_2c_3\cdots n_n| = \{a_1'A_1 + b_1'B_1 + \cdots + n_1'N_1\}$$
$$+ \{a_1''A_1 + b_1''B_1 + \cdots + n_1''N_1\}.$$

The first bracket is clearly the Determinant $|a_1'b_2c_3\cdots n_n|$, the second is $|a_1''b_2c_3\cdots n_n|$; hence, it is plain that

$$|(a_1' + a_1'')b_2c_3\cdots n_n| = |a_1'b_2c_3\cdots n_n| + |a_1''b_2c_3\cdots n_n|.$$

In this way one Determinant may always be expressed as the *sum* of two *part*-Determinants which have all their rows (or columns) the same as in the whole Determinant, but one pair of corresponding ones, while the *sum* of any two corresponding elements in this pair equals the corresponding element in the corresponding row (or column) in the whole Determinant.

It is now clear that any Determinant may be broken up into 3, or, indeed, into any number of part-Determinants, each element of a row being supposed made up of 3, or any number of parts, thus:

$$a_1 = a_1' + a_1'' + a_1''', \quad b_1' + b_1'' + b_1''', \quad \text{etc.}$$

If every row (or column) be broken up into parts this way, a part-Determinant may be formed by taking for a 1st column any part-column of the 1st column, for a 2d column any part-column of the 2d column, and so on throughout. Hence the *total number* of part-Determinants will be the *product of the numbers* of part-columns for all the columns.

Evaluation of Determinants.

10. In the Determinant $|a_1b_2c_3\cdots n_n|$ add the 2d column to the 1st, each element to its correspondent; we get

$$|(a_1+b_1)b_2c_3\cdots n_n|=|a_1b_2c_3\cdots n_n|+|b_1b_2c_3\cdots n_n|.$$

The 2d Determinant on the right has two identical columns of b's, hence its value is 0; the 1st on the right is the original one; hence the new Determinant on the left equals the old one. If instead of adding the 2d column we had added its m-fold, we should have got m times the 2d Determinant on the right, which would still be 0; plainly, too, the reasoning about any other pair of columns or about rows would be the same. Hence, the *value of a Determinant is not changed by adding to each of its elements in one row* (or column) *any fixed multiple of the correspondent elements in any other row* (or column).

This theorem furnishes a ready method of reducing the degree of a Determinant. For by proper additions all the elements of a row (or column), say the 1st, may be made 0 but one; then the whole Determinant will be equal to this one multiplied by its co-factor, for we shall have

$$\begin{aligned}|a_1b_2c_3\cdots n_n|&=a_1A_1+0\cdot B_1+0\cdot c_1+\cdots+0\cdot N_1\\&=a_1|b_2c_3\cdots n_n|.\end{aligned}$$

The degree of this co-factor is clearly one less than the degree of the original Determinant; by repeating this process the degree of the Determinant may be brought down to 2 or even to 1.

Thus far the reasoning has been so closely connected and withal so simple that it has been deemed best not to interrupt it in any way. The following examples will amply illustrate all the foregoing.

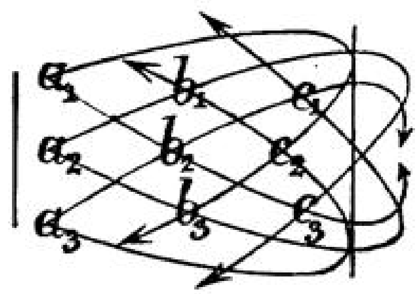
1. 4 6 3 1 5 2 7. In this permutation the inversions are: 43, 41, 42, 63, 61, 65, 62, 31, 32, 52, ten in number, the permutation is even. *b c a e f g d*. Here the inversions are: *ba*, *ca*, *ed*, *fd*, *gd*, five in number, the permutation is odd. On exchanging 6 and 2, the number of inversions falls to five, the permutation becomes odd; on exchanging *a* and *f*, the number of inversions rises to eight, the permutation becomes even.

2. The permutations of 1 2 3 are, in pairs: 123, 132; 213, 312; 231, 321; one of each pair is even, one odd. Which?

3. The Determinant of 2d degree $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$

4. $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1.$$

The number of terms is $3! = 6$, so we write off the combination $a\,b\,c$ six times and suffix the subscripts permuted. The numbers of inversions in the permutations are resp.: 0, 1, 2, 1, 2, 3, and the signs are prefixed accordingly. A simple mechanical rule for calculating the Determinant of 3d degree as shown in this diagram:



The arrows turning up are drawn through the + combinations; those turning down, through the − ones. But this does *not* hold for higher degrees.

5. $$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} - d_1 \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}.$$

6. $$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

7. $$\begin{vmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1+\gamma_1 \\ a_2+\alpha_2 & b_2+\beta_2 & c_2+\gamma_2 \\ a_3+\alpha_3 & b_3+\beta_3 & c_3+\gamma_3 \end{vmatrix}$$
$$= |a_1 b_2 c_3| + |a_1 b_2 \gamma_3| + |a_1 \beta_2 c_3| + |a_1 \beta_2 \gamma_3| + |\alpha_1 b_2 c_3|$$
$$+ |\alpha_1 b_2 \gamma_3| + |\alpha_1 \beta_2 c_3| + |\alpha_1 \beta_2 \gamma_3|.$$

8. $$\begin{vmatrix} a_1+mb_1 & b_1 & c_1 \\ a_2+mb_2 & b_2 & c_2 \\ a_3+mb_3 & b_3 & c_4 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} mb_1 & b_1 & c_1 \\ mb_2 & b_2 & c_2 \\ mb_3 & b_3 & c_3 \end{vmatrix}$$
$$= |a_1 b_2 c_3| + m\,|b_1 b_2 c_3| = |a_1 b_2 c_3|.$$

9. $$\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} = b_1b_2c_3 - b_1b_3c_2 + b_2b_3c_1 - b_2b_1c_3 + b_3b_1c_2 - b_3b_2c_1 = 0.$$

10. $$\begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} = a_1c_2b_3 - a_1c_3b_2 + a_2c_3b_1 - a_2c_1b_3 + a_3c_1b_2 - a_3c_2b_1$$
$$= -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

11. $$\begin{vmatrix} 4 & 5 \\ 6 & 8 \end{vmatrix} = 32 - 30 = 2\begin{vmatrix} 2 & 5 \\ 3 & 8 \end{vmatrix} = 2(16-15) = 2\begin{vmatrix} 4 & 5 \\ 3 & 4 \end{vmatrix}$$
$$= 2(16-15) = 2.$$

12. $$\begin{vmatrix} 4 & 5 & 2 \\ 6 & 8 & 10 \\ 2 & 1 & 3 \end{vmatrix} = 96 + 12 + 100 - 32 - 40 - 90 = 46.$$

Also

$$\begin{vmatrix} 4 & 5 & 2 \\ 6 & 8 & 10 \\ 2 & 1 & 3 \end{vmatrix} = 2\begin{vmatrix} 2 & 5 & 2 \\ 3 & 8 & 10 \\ 1 & 1 & 3 \end{vmatrix} = 2\begin{vmatrix} 0 & 3 & -4 \\ 0 & 5 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 2\begin{vmatrix} 3 & -4 \\ 5 & 1 \end{vmatrix}$$
$$= 2(3+20) = 46.$$

13. $\begin{vmatrix} 2 & 3 & 5 & 7 \\ 5 & 4 & 2 & 6 \\ 7 & 2 & 3 & 8 \\ 3 & 4 & 2 & 5 \end{vmatrix} = \begin{vmatrix} -1 & -1 & 3 & 2 \\ 5 & 4 & 2 & 6 \\ 7 & 2 & 3 & 8 \\ 3 & 4 & 2 & 5 \end{vmatrix} = \begin{vmatrix} -1 & -1 & 3 & 2 \\ 0 & -1 & 17 & 16 \\ 0 & -5 & 24 & 22 \\ 0 & 1 & 11 & 11 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 17 & 16 \\ 5 & 24 & 22 \\ -1 & 11 & 11 \end{vmatrix} = \begin{vmatrix} 1 & 17 & 16 \\ 0 & 79 & 77 \\ 0 & 28 & 27 \end{vmatrix} = \begin{vmatrix} 2 & 77 \\ 1 & 27 \end{vmatrix} = -23.$$

14. Reckon $\begin{vmatrix} k & h & g \\ h & j & f \\ g & f & c \end{vmatrix}$, and find the co-factors K, H, G, F, C, J.

15. Reckon $\begin{vmatrix} 5 & 3 & 2 \\ 4 & 7 & 8 \\ 9 & 2 & 6 \end{vmatrix}, \begin{vmatrix} 3 & -2 & 5 \\ 6 & 4 & -3 \\ 2 & -7 & 6 \end{vmatrix}, \begin{vmatrix} 8 & 3 & -2 \\ 5 & -7 & 6 \\ 4 & 9 & -5 \end{vmatrix}, \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

16. Bring the equation $\begin{vmatrix} x & y & 1 \\ \dfrac{r_1x_2 - r_2x_1}{r_1 - r_2} & \dfrac{r_1y_2 - r_2y_1}{r_1 - r_2} & 1 \\ \dfrac{r_2x_3 - r_3x_2}{r_2 - r_3} & \dfrac{r_2y_3 - r_3y_2}{r_2 - r_3} & 1 \end{vmatrix} = 0$

into the form $\begin{vmatrix} r_1 & r_2 & r_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} x - \begin{vmatrix} r_1 & r_2 & r_3 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} y + \begin{vmatrix} r_1 & r_2 & r_3 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{vmatrix} = 0.$

Multiplication of Determinants.

11. An interesting case of Art. 9, as illustrated in Example 7, is this:

$$\begin{vmatrix} a_1\alpha_1+b_1\beta_1+c_1\gamma_1+\cdots+n_1\nu_1, & a_1\alpha_2+b_1\beta_2+\cdots+n_1\nu_2, & \cdots\cdots\cdots, & a_1\alpha_n+b_1\beta_n+\cdots+n_1\nu_n \\ a_2\alpha_1+b_2\beta_1+c_2\gamma_1+\cdots+n_2\nu_1, & a_2\alpha_2+b_2\beta_2+\cdots+n_2\nu_2, & \cdots\cdots\cdots, & a_2\alpha_n+b_2\beta_n+\cdots+n_2\nu_n \\ \vdots & \vdots & \vdots\ \vdots\ \vdots & \vdots \\ a_n\alpha_1+b_n\beta_1+c_n\gamma_1+\cdots+n_n\nu_1, & a_n\alpha_2+b_n\beta_2+\cdots+n_n\nu_2, & \cdots\cdots\cdots, & a_n\alpha_n+b_n\beta_n+\cdots+n_n\nu_n \end{vmatrix}$$

We notice here that the Greek and the Roman letters enter this Determinant in the same way: the Greek appear in the columns as the Roman do in the rows, and *vice versa*. Again, this whole Determinant, which we may write ΔGR, clearly breaks up into the sum of n^n part-Determinants; for any of the 1st n part-columns may be combined with any one of the 2d n part-columns, and so on. But all but n ! of these part-Determinants *vanish;* for, if the Roman letter of one part-column be the same as the Roman letter of any other, then on setting out the Greek factor the part-determinant will have two columns identical, and vanishes by Art. 6. Accordingly, to get part-Determinants not $= 0$, we must pick out each time all the Roman letters for our part-Determinant, one for each part-column; hence, on setting out the Greek factors, the part-columns become the columns of the Determinant of Roman letters, $|a_1 b_2 c_3 \dots n_n|$, which we may write ΔR; hence, our part-Determinant contains as a factor a determinant that can differ from ΔR only in the order of its columns, i.e., only in sign; hence, being a factor of each part-Determinant, ΔR is a factor of the whole Determinant. Hence, since the Greek and the Roman letters enter ΔGR alike, ΔG is also a factor of ΔGR. Now the terms of the product $\Delta G \cdot \Delta R$ are of $2n$th degree, and so are the terms of ΔGR; also the number of terms both in ΔGR and in $\Delta G \cdot \Delta R$ is the same, $n!\,n!$ Hence, ΔGR can differ from $\Delta G \cdot \Delta R$, if at all, only in sign; i.e., $\Delta GR = \pm \Delta G \cdot \Delta R$. To decide as to the sign, consider the product of the diagonal terms; it is + in $\Delta G \cdot \Delta R$; the like term is also + in ΔGR. For it is got by taking the 1st part-column of the 1st column, the 2d of the 2d, etc.; the factors set out are $\alpha_1 \beta_2 \gamma_3 \dots \nu_n$, and the order of the columns of Roman letters is natural, as in ΔR; and in ΔR the diagonal $a_1 b_2 c_3 \dots n_n$ is +, as is the diagonal term in every Determinant. Now one pair of corresponding terms being like-signed in $\Delta G \cdot \Delta R$ and ΔGR, all are like-signed; i.e., $\Delta G \cdot \Delta R = \Delta GR$.

ILLUSTRATION:

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1, & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2, & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1, & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2, & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1, & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2, & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

$$= \alpha_1\beta_2\gamma_3 |a_1 b_2 c_3| + \alpha_1\gamma_2\beta_3 |a_1 c_2 b_3| + \beta_1\gamma_2\alpha_3 |b_1 c_2 a_3|$$

$$+ \beta_1\alpha_2\gamma_3 |b_1 a_2 c_3| + \gamma_1\alpha_2\beta_3 |c_1 a_2 b_3| + \gamma_1\beta_2\alpha_3 |c_1 b_2 a_3|.$$

Here the inversions in the order of the columns of Roman letters are those got by permuting those letters; if we restore them to natural order, the indices, which are now in natural order, will be permuted as were the letters; also, restoring them will change or not change the sign of the term according as the number of inversions is even or odd; but the inversions in the order of Roman letters are the same as in the order of the Greek; hence, the Greek products are + or − according as the permutations of the letters are even or odd; hence, the sum of the Greek products is the Determinant of the Greek letters. As no reference has been made to the number of letters, 3, this proof is quite as general as the one already given. Actually bringing the Roman letters into natural order, we get

$$\Delta GR = a_1\beta_2\gamma_3\Delta R - a_1\beta_3\gamma_2\Delta R + a_3\beta_1\gamma_2\Delta R - a_2\beta_1\gamma_3\Delta R$$
$$+ a_2\beta_3\gamma_1\Delta R - a_3\beta_2\gamma_1\Delta R = \Delta G \cdot \Delta R.$$

12. If $a_1 = A_1$, $a_2 = A_2$, ..., $\beta_1 = B_1$, ..., the Greek letters being the co-factors of the corresponding Roman letters, then, in ΔGR all the elements vanish but the diagonal ones, by Art. 8, and these are each ΔR; hence ΔGR has but one term, the diagonal term, and that is ΔR^n. Writing Δ for ΔR and Δ for ΔG, and remembering $\Delta GR = \Delta G \cdot \Delta R$, we have

$$\Delta \cdot \Delta = \Delta^n \quad \text{or} \quad \Delta = \Delta^{n-1};$$

the Determinant of the co-factors of the elements of a Determinant of n*th degree* is the $(n-1)$th *power* of this Determinant.

In particular, if $n = 3$, $\Delta = \Delta^2$.

If the co-factor of A_1 *be* a_1', and so on, and if the Determinant of the co-factors a_1', etc., of the co-factors A_1, etc., be written Δ', then

$$\Delta' = \Delta^{n-1} = (\Delta^{n-1})^{n-1} = \Delta^{n^2-2n+1}.$$

Now if we multiply $a_1, b_1, \ldots, a_2, \ldots$, each element of Δ by Δ^{n-2} or $\Delta : \Delta$, it will be the same as to multiply Δ by $\Delta^{(n-2)n}$, since it multiplies each row (or column) by Δ^{n-2}; the resulting Determinant is then Δ^{n^2-2n+1}, or is Δ'. Hence, we may conclude that $a' = a \cdot \Delta^{n-2}$, a proposition we have not space to prove more rigorously.

In particular, if $n = 3$, $a' = a \cdot \Delta$.

Applications.

13. If there be given a system of n Equations containing n Unknowns in first degree only, it is possible by successive elimination to get *one* Equation with *one* Unknown, whence this Unknown may be found; then, by substitution, all the others may be found, one by one. But this process is very tedious. The theorems of Art. 8 furnish a direct solution of the problem.

Call the Unknowns $u_1, u_2, \cdots, u_n$; then

$$\begin{array}{l} a_1u_1 + b_1u_2 + \cdots \;\cdots\; \cdots + n_1u_n = k_1 \\ a_2u_1 + b_2u_2 + \cdots \;\cdots\; \cdots + n_2u_n = k_2 \\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots \qquad \vdots \\ a_nu_1 + b_nu_2 + \cdots \;\cdots\; \cdots + n_nu_n = k_n. \end{array}$$

If the Equations be multiplied by $A_1, A_2, \cdots A_n$, in turn and summed, where $A_1, A_2, \cdots, A_n$ are the co-factors of $a_1, a_2, \cdots a_n$, in the determinant of the coefficients $|\, a_1b_2c_3 \cdots n_n \,|$, the coefficient of u_1 in the sum will be

$$a_1A_1 + a_2A_2 + \cdots + a_nA_n, = |\, a_1b_2c_3 \cdots n_n \,|, \quad \text{by Art. 8,}$$

while the coefficient of any other u, as u_2, will be

$$b_1A_1 + b_2A_2 + \cdots + b_nA_n, = 0, \qquad \text{by Art. 8,}$$

and the absolute term in the sum will be

$$k_1A_1 + k_2A_2 + \cdots + k_nA_n = |\, k_1b_2c_3 \cdots n_n \,|;$$

or, $$|\, a_1b_2c_3 \cdots n_n \,| u_1 = |\, k_1b_2c_3 \cdots n_n \,|,$$

whence $$u_1 = \frac{|k_1b_2c_3 \cdots n_n|}{|a_1b_2c_3 \cdots n_n|}.$$

Note that the denominator is the Determinant of the coefficients in order; the numerator is got from the denominator by replacing the column of coefficients of the Unknown in question by the column of absolutes.

Plainly the reasoning holds alike for all the Unknowns.

14. If the absolutes, the k's, be all 0, then the numerator of the value of each Unknown, of each u, has a column of 0's, hence itself is 0; hence the value of each u must be 0 *unless the denominator also be* 0; when the k's are 0, the Equations are *homogeneous* in the u's; hence

The condition that n homogeneous equations of 1st degree in n Unknowns may consist, is that *the Determinant of the coefficients of the Unknowns vanish.*

Where the Equations are homogeneous in the u's, each may be divided by one of the u's, say u_n; the quotients of the u's being considered as *new* Unknowns, *new* u's, there are n Equations and only $(n-1)$ Unknowns, while the coefficients of u_n are absolutes; the condition of consistence of the Equations is unchanged; hence

The condition that n Equations among $(n-1)$ Unknowns in 1st degree may consist, is that the *Determinant of the coefficients and the absolutes vanish.*

15. Often it is required to eliminate one Unknown between two Equations of higher degree in that Unknown; or, what is the same, to find what relation must hold among the coefficients in the two Equations, if the two are to consist, i.e., hold for the same values of the Unknown. An example will make this clear.

Given $ax^3+bx^2+cx+d=0$ and $ex^2+fx+g=0$; find the condition that these Equations consist, i.e., that the roots of the 2d be also roots of the 1st.

When the first Equation is satisfied, so is this:

$$ax^4+bx^3+cx^2+dx=0;$$

when the second is, so are these:

$$ex^3+fx^2+gx=0 \quad \text{and} \quad ex^4+fx^3+gx^2=0;$$

hence, when both are satisfied, so are these five:

$$\begin{aligned}
0\cdot x^4 + ax^3 + bx^2 + cx + d &= 0,\\
a\cdot x^4 + bx^3 + cx^2 + dx + 0 &= 0,\\
0\cdot x^4 + 0\cdot x^3 + ex^2 + fx + g &= 0,\\
0\cdot x^4 + e\cdot x^3 + fx^2 + gx + 0 &= 0,\\
e\cdot x^4 + fx^3 + gx^2 + 0\cdot x + 0 &= 0.
\end{aligned}$$

Here are 5 Equations containing 4 Unknowns: x^4, x^3, x^2, x; by Art. 14 they consist when and only when

$$\begin{vmatrix} 0 & a & b & c & d \\ a & b & c & d & 0 \\ 0 & 0 & e & f & g \\ 0 & e & f & g & 0 \\ e & f & g & 0 & 0 \end{vmatrix} = 0.$$

Clearly this method is always applicable. Be one Equation of nth degree, the other of $(n+d)$th degree; by multiplying the 1st by x d times successively we raise it to the $(n+d)$th degree, and get in all $d+2$ Equations and $n+d$ Unknowns; then by multiplying each of the two Equations of $(n+d)$th degree by x we get two more Equations and one more Unknown, the next higher power of x, x^{n+d+1}; by $n-1$ such successive multiplications we get in all $2n+d$ Equations and $2n+d-1$ Unknowns; then Art. 14 is to be applied.

Under the hands of British and Continental masters the Theory of Determinants has been of late years built up to colossal size and applied to almost every branch of mathematics; in fact, it has become well-nigh indispensable to higher research. An excellent English work is Muir's *Theory of Determinants*.

EXERCISES.

1. Solve the systems of equations:

$$3x+4y-5z=7,\quad 2x-3y-4z=9,\quad 4x-5y+z=8;$$

$$x-y+2z+5v=10,\quad 2x+3y-z+v=7,$$

$$4y-3x+8z-2v=5,\quad 3z-2y-5x+7v=3.$$

2. Do these systems of equations consist?

$$5x - 3y = 7, \quad 8x + 5y = 9, \quad 3x - 2y = -4;$$

$$2x + 3y - 4z = 5, \quad 5x - 2y + 3z = 7, \quad x + 5y - 5z = -8,$$

$$4x + 3y - 3z = 2.$$

CO-ORDINATE GEOMETRY.

PART I. THE PLANE.

CHAPTER I.

INTRODUCTORY: FIRST NOTIONS.

Function and Argument.

1. In a table of logarithms are found two series of corresponding values: one of natural numbers and one of logarithms. Given any number, we may find from such table *a* corresponding logarithm; given any logarithm, we may find *the* corresponding number. Like may be said of a table of natural sines: given any number (expressed commonly in degrees), we can find *the* corresponding sine; given any sine, we can find *a* corresponding number. Such tables are calculated to a greater or less degree of exactness by rules or formulæ; other like tables are found in works on Physics, calculated, however, not by rule, but by experiment. From such a table we may find (within certain limits) *e.g.* the tension of saturated vapor of water for every degree of temperature, and conversely; but we know no rule to calculate one from the other.

Two magnitudes, such that to any value of one corresponds a value of the other, are called **functions** *of each other.*

Such are a number and its logarithm; a number and its sine; the surface or volume of a sphere and its radius; the velocity of a wave-motion, as of sound, and the elasticity of the medium; the density of pure water and its temperature; etc.

2. As is seen, in Physics the interdependent magnitudes called *functions* in general define physical states, and the nature of their interdependence cannot generally be stated in a formula or rule; in Mathematics the interdependents are empty forms, symbols: x, y, z, a, b, c, and the nature of their interdependence is expressed by a *formula* or *equation*.

Take, for example, the Eq.,* $2x+3y=12$.

Assigning arbitrary values to x *respectively* y, we reckon the corresponding values of y *resp.* x by the *formulæ* (rules):

$$y=\overline{12-2x}:3 \qquad \text{resp.} \qquad x=\overline{12-3y}:2.$$

A series of *pairs* of corresponding values is

$$(x,y)=|\,(-3,6)\,;\ (-2,\tfrac{16}{3})\,;\ (-1,\tfrac{14}{3})\,;\ (0,4)\,;\ (1,\tfrac{10}{3})\,;\ (2,\tfrac{8}{3})\,;\ (3,2)\,;\ \cdots\,|.$$

In the unsolved Eq., $2x+3y=12$, x and y stand on precisely like footing: each is an *implicit* function of the other, but in the Eq. solved as to one of them, say y, thus

$$y=\overline{12-2x}:3,$$

they no longer stand on like footing; contrariwise, this Eq. gives a *rule* for reckoning the value of y for any assigned value of x, but *not* conversely. *That symbol to which we assign arbitrary values is called the* **Argument**; the symbol whose values are reckoned is called specifically the *function;* if the Eq. be solved as to any symbol, that symbol is called an *explicit* function.

ILLUSTRATION. $x^2+y^2=25$: here x and y are each an *implicit function* of the other; solved,

$y=\pm\sqrt{25-x^2}$: here y is an *explicit function* of the *argument* x.

$x=\pm\sqrt{25-y^2}$: here x is an *explicit function* of the *argument* y.

---

* Short for *Equation*.

N. B. Though the relation between two symbols be accurately expressed by an Eq., yet it is not in general possible to state a rule for reckoning one through the other, since the general Eq. of degree higher than the fifth has not yet been solved. None the less, the symbol to which we suppose arbitrary values assigned is still called the argument; that, as to which we suppose the Eq. solved, the function.

3. If in the Eq. connecting two symbols each be operated upon by a *finite* number of algebraic operations, additions, multiplications, involutions, and their inverses, then is each called an *algebraic function* of the other: but if the number of such operations upon either be *infinite*, then are they called *transcendental functions* of each other; as

$$y = \sin x \equiv x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \frac{x^7}{7\,!} + \cdots \equiv \sum_{n=0}^{n=\infty} (-1)^n \frac{x^{2n+1}}{2n+1\,!}$$

$$y = e^x \equiv 1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \frac{x^4}{4\,!} + \cdots \equiv \sum_{n=0}^{n=\infty} \frac{x^n}{n\,!}.$$

More important for us is this distinction: when to *one* value of the argument correspond *one*, *two*, *three*, resp. *many* values of the function, the latter is called a *one-*, *two-*, *three-*, resp. *many*-valued function of the argument; as

$2x + 3y = 12$: x and y each *one*-valued functions of the other;

$x^2 + y^2 = r^2$: x and y each *two*-valued functions of the other;

$y^2 = 4px$: x a *one*-valued function of y, y a *two*-valued function of x;

$y = \cos x \equiv \sum (-1)^n \frac{x^{2n}}{2n\,!}$: y a *one*-valued function of x, x an *infinitely many*-valued function of y (since, if $x = r$ be any root of the Eq. for any assigned value of y, then is also $x = \pm 2n\pi \pm r$ a root, n being any natural number).

EXERCISES.

What functions of each other are x and y in

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;\quad y^3-3axy+x^3=0;\quad (x^2+y^2)^3=4a^2x^2y^2;$$

$$y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)?$$

From the last Eq. express x as an explicit function of y.

4. To different values of the argument correspond in general different values of the function; if the difference between two argument-values be very small, in general the difference between the corresponding function-values will be very small: as the argument changes gradually, so does the function. Be now $x_2-x_1\equiv\Delta x$ the difference between two argument-values, $y_2-y_1\equiv\Delta y$ the difference between the corresponding function-values; if then, by taking $\Delta x<\sigma$, we can make $\Delta y<\sigma'$ for all values of $\Delta x<\sigma$, where the σ's mean magnitudes small at will, then is y called a **continuous** *function* of x.

ILLUSTRATIONS. $2x+3y=12$. If (x_1,y_1), (x_2,y_2) be two *pairs* of corresponding values,

then $\quad 2x_1+3y_1=12,\quad$ and $\quad 2x_2+3y_2=12;$

whence $\quad x_2-x_1=-\frac{3}{2}(y_2-y_1),\quad$ or $\quad \Delta y=-\frac{2}{3}\Delta x.$

The sign $-$ shows that, as either x or y increases, the other decreases; but we have here to deal only with the absolute (or signless) values of the differences Δx and Δy, and since their ratio 2 : 3 is finite, clearly we can make and keep either small at will by making and keeping the other small at will. This is so for every finite value of x or y; hence, each is a *continuous* function of the other for $-\infty<x<\infty$, $-\infty<y<\infty$.

Like may be shown of the functions $y=\sin x$, $y=\cos x$, but *not* of their quotient

$$y=\tan x\equiv\frac{\sin x}{\cos x}.$$

For $x=-\frac{\pi}{2}+h$, h very small, $\sin x=-1$ nearly, $\cos x$ is + and = 0 nearly; hence, $y\equiv\tan x$ is very great and negative. As h increases toward $\frac{\pi}{2}$, $\sin x$ increases toward -0 (i.e., nears 0 from the − side), $\cos x$ increases toward $+1$, $\tan x$ increases toward -0 gradually; as h nears π, x nears $\frac{\pi}{2}$, $\sin x$ nears $+1$, $\cos x$ nears $+0$, $\tan x$ becomes + and very great, but changes throughout gradually with x. For $x=\frac{\pi}{2}-\sigma$, σ small at will and +, $\tan x$ is very great and +; for $x=\frac{\pi}{2}+\sigma$, $\cos x$ changes from the + to the − side of 0, $\sin x$ changes not, $\tan x$ changes from being very great and + to being very great and −. Hence, for $-\frac{\pi}{2}<x<\frac{\pi}{2}$, $\tan x$ is a *continuous* function of x; but for $x=\frac{\pi}{2}$, $\tan x$ is a **discontinuous** function of x: if $x_1<\frac{\pi}{2}$, $x_2>\frac{\pi}{2}$, we can *not* make $y_2-y_1\equiv\tan x_2-\tan x_1\equiv\Delta y$ small at will by making $x_2-x_1\equiv\Delta x$ small at will; as x passes through the value $x=\frac{\pi}{2}$, $\tan x$ springs from $+\infty$ to $-\infty$. The value $x=\frac{\pi}{2}$ is called a *point of discontinuity* for the function $y\equiv\tan x$. Since π is the period of the tangent, i.e., $\tan x=\tan(x\pm\pi)$, $x=(\pm 2n+1)\frac{\pi}{2}$ is also a point of discontinuity.

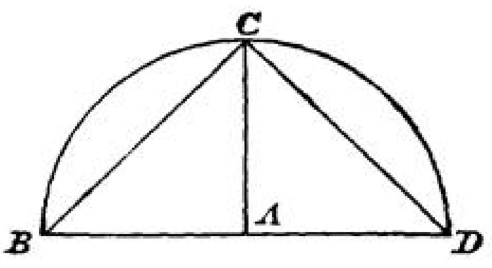
EXERCISE.

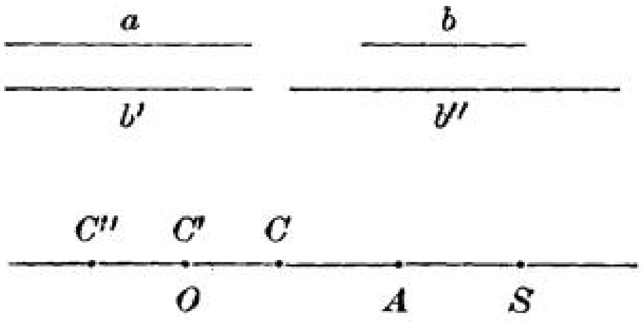
Show that $y=c\cdot\dfrac{e^{\frac{1}{x-a}}-1}{e^{\frac{1}{x-a}}+1}$ is discontinuous for $x=a$.

HINT. As x nears a, increasing, y nears $-c$; as x nears a, decreasing, y nears $+c$; as x passes through a, increasing resp. decreasing, y springs from $-c$ to $+c$ resp. from $+c$ to $-c$.

Geometric Representation of Magnitudes.

5. Any measurable magnitude may be represented by a number, called its *metric number:* the ratio of the magnitude to an assumed unit-magnitude; a number may be represented, or pictured, by some definite part of a line, some arbitrary *tract* (i.e., definite part of a right line) being taken to picture 1. Thus, if AB picture 1, BD will picture 2, BC will picture $\sqrt{2}$,





BCD will picture π. Instead of the tract whose metric number is a may be said briefly the tract a. On the ray OD lay off from O any number of tracts picturing as many (positive) numbers. To picture the sum of any two numbers, a and b, lay off a tract s equal to the sum of the tracts a and b. To picture the difference of two numbers, a and b, lay off a tract d such that the sum of the tracts b and d shall equal the tract a. To do this, lay off from the end A of tract a, toward O, i.e., counter to the direction a was laid off in, a tract b to C; then is OC the sought tract d. Three cases may arise:

(1) $a > b$, then the point C falls between O and A;

(2) $a = b'$, then the point C' falls on O, the difference d is 0;

(3) $a < b''$, then the point C'' falls beyond O leftward; but then the difference $a - b''$ or d is negative; hence negative numbers are pictured by tracts laid off counter to the direction in which are laid off tracts picturing positive numbers. But in this way was laid off a positive tract, to subtract it; hence, to

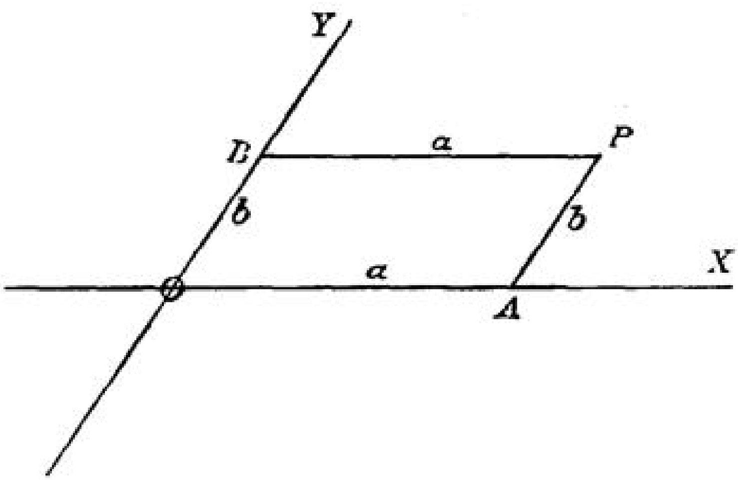
add a negative tract, subtract an equal positive one; hence, too, to subtract a negative tract, add an equal positive one.

N.B. To add to a tract, or to subtract from it, we lay off from its *end:* forwards, from it, to add a positive, or subtract a negative, tract; backwards on it, to subtract a positive, or add a negative, tract. Like reasoning and conclusions hold for angles and arcs.

6. Assuming, then, any RL. (short for right line), we may picture all real numbers by tracts laid off on it rightward and leftward from any assumed point O. As all such tracts have

the same beginning, O, each is fully defined by its *end.* Accordingly, not only the tract OA by its length and direction, but its end, the point A, by its position as to O, pictures the number a; so, every point on the RL. OD pictures some real number, the RL. itself pictures the whole of real numbers.

If we take two RLs., as OX, OY, intersecting under any $\measuredangle\ \omega$, each will picture by its points the whole of real numbers, the section O naturally taken to picture zero in each picture. *Any point P in the plane of the RLs. will picture* then not simply one number but **a pair of numbers**, as (a, b); for its distances from the zero-point O, measured along these RLs., are a and b.

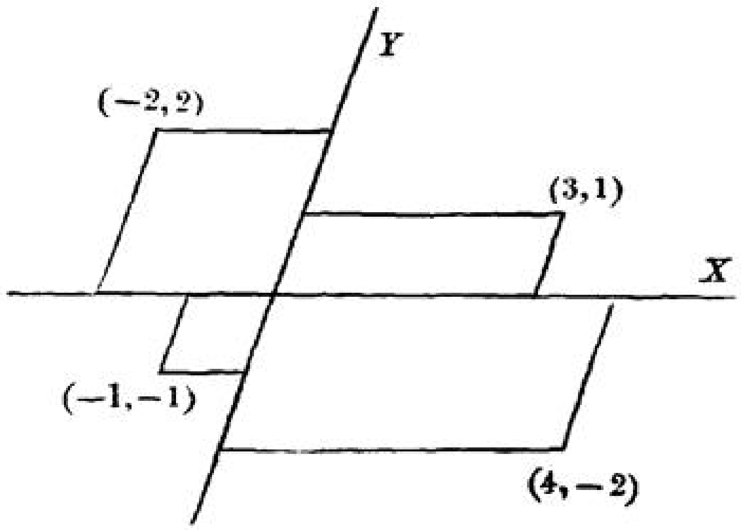


N.B. The choice of RLs. and of positive and negative directions is arbitrary; it is common to treat rightward and upward as +, leftward and downward as

EXERCISES.

In the figure, the corner-points picture the pairs of numbers bracketed by them.

1. Find the points that picture these pairs: $(1, 3)$, $(-3, -1)$, $(-2, 1)$, $(4, -5)$, $(0, \frac{5}{2})$, $(0, 5)$, $(5, 0)$, $(6, 0)$, $(0, 0)$.



2. Where are all points picturing pairs whose first term is 0; i.e., of the form $(0, b)$?

3. Of the form $(a, 0)$?

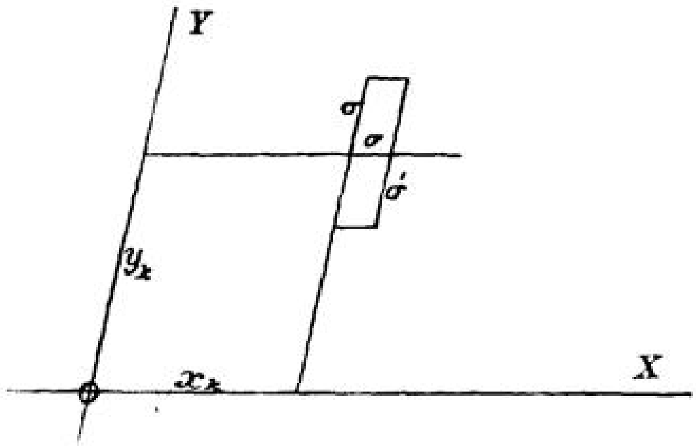
4. Where are all pictures of pairs whose terms are equal; i.e., of the form (a, a)?

5. Of the form $(a, -a)$?

6. Where are all pictures of pairs whose first terms are all alike, second terms unlike?

7. Whose second terms are all alike, the first unlike?

7. Be now $f(x, y) = 0$ (read: the f-function of x and y equals 0) any Eq. determining x and y as functions of each other, and suppose the functions continuous and one-valued. Be (x_1, y_1), (x_2, y_2), . . . (x_n, y_n) pairs of corresponding values; i.e., be $f(x_1, y_1) = 0$, $f(x_2, y_2) = 0, \ldots f(x_n, y_n) = 0$. Each pair is pictured by a point in the plane of OX and OY. Suppose the x's subscribed in the order of size, thus: $x_1 < x_2 < x_3 < \cdots < x_n$. By taking consecutive values of x very close to each other, we make the consecutive values of y very close to each other; this series of pairs of values of x and y will then be pictured by a series of points consecutively very close to each other. Now, it is true, however close together we may heap these points, we can never make out of them a line. But if the function be continuous, by taking $x_{k+1} - x_k = \sigma$ we can make $y_{k+1} - y_k = \sigma'$, and for every value of x between x_{k+1} and x_k the corresponding value of y will differ from y_k by $< \sigma'$, positively or negatively; i.e., all points picturing pairs of values for x between x_{k+1} and x_k will lie in the double parallelogram whose sides are σ and $2\sigma'$. Hence all points picturing pairs of values that satisfy the Eq. $f(x, y) = 0$ lie in a series of contiguous double parallelograms, whose sides are small at will; so, too, are the diagonals of the single parallelograms.



The train of diagonals (that one of each couple being taken that joins two picturing points) will form a polygon (open or closed) whose vertices are picturing points, and by taking the values of x, and therefore of y, ever closer and closer together, we make this polygon near as its limit a definite curve, of which each diagonal is a chord. Every pair of values satisfying the Eq. $f(x, y) = 0$ is pictured by a point on this curve; and conversely, every point of this curve pictures a pair of values satisfying the Eq. $f(x, y) = 0$. Hence,

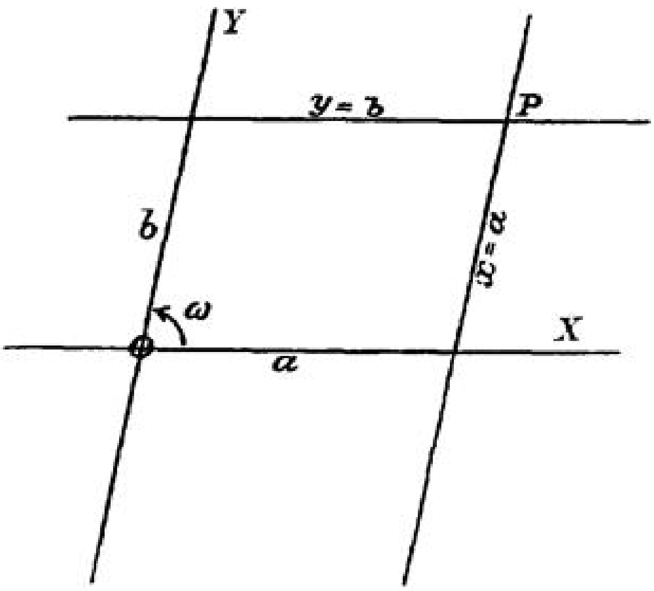
A geometric picture of the Eq. $f(x, y) = 0$ *is a plane curve.* We say indifferently the Eq. $f(x, y) = 0$ and the curve $f(x, y) = 0$.

N.B. The curve breaks up in case: of a many-valued function, into many branches; of a discontinuous function, into distinct parts. But the reasoning needs but slight change. If some other than a RL. be used to picture a series of numbers, like reasoning and conclusions hold.

Determination of Position on a Surface.

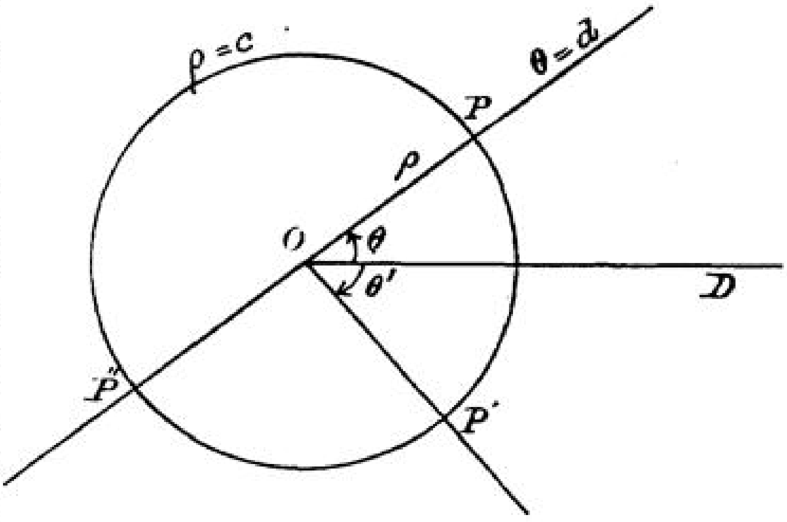
8. We say of a surface: it is doubly extended, or has two dimensions, meaning that two independent measurements are necessary and sufficient to fix any point on the surface. Thus we know any place on the earth's surface, knowing its latitude and longitude. In this familiar example we suppose the surface covered with a double system of lines: half-meridians and perpendicular parallel small circles. Through any point of the surface passes one and only one half-meridian, one and only one parallel; hence, knowing any point, we know its meridian and parallel. Also, any half-meridian cuts any parallel in one and only one point; hence, knowing any meridian and any parallel, we know their junction-point. The parallels are named from their angular distance from the mid-parallel (equator); the half-meridians from their angular distance from an assumed fixed half-meridian.

Likewise we may think a plane covered with a double system of (say parallel right) lines, making any angle ω with each other. Through any point in the plane passes one and only one line of each system; hence, knowing any point, we know what pair of lines meet in it. Conversely, any pair of lines meet in one and only one point of the plane; hence, knowing any pair of lines, we know their junction-point. We name and know each line of a system by its distance from an assumed fixed line of the system (measured on any line of the other system). If this measurement be rightward or up, the metric number of the distance is marked $+$; if leftward or down, $-$. The assumed fixed RLs., as OX, OY, are called Co-ordinate Axes, or axes of X and Y, or X- and Y-axes. The angle ω, reckoned from the $+X$- to the $+Y$-axis, is called the co-ordinate angle; for $\omega = 90°$, the axes are rectangular; otherwise, oblique. The junction-point, O, of the axes is called the Origin.



9. A point is fixed as the junction of a pair of parallels, or co-ordinate lines; conversely, a pair of co-ordinate lines are fixed as meeting in a point. The distance from the origin O at which a co-ordinate line cuts the X- resp. Y-axis is called its intercept on that axis. Such intercepts are denoted by the symbols x resp. y. If a be the metric number of the x-intercept of any parallel to the Y-axis, then is this parallel known completely from the Eq. $x=a$, which is therefore called its Eq. So $y=b$ is the Eq. of a parallel to the X-axis making an intercept b on the Y-axis. The junction-point of this pair is known completely from the two Eqs. $x=a, y=b$, which are therefore called the Eqs. of the point. The point itself is spoken of as the point (a, b) or as $P(a, b)$; a is called the **abscissa** or **x** of the point, b, its **ordinate** or **y**; a and b, its **co-ordinates** or its **x** and **y**. We may now convert the proposition of Art. 6, thus: *Any point in a plane may be represented by a* **pair of numbers**: *the rectilinear* **co-ordinates** *of the point.*

10. We may think the plane covered with some other double system of lines: as a system of rays from the centre of a system of concentric circles. Each point is fixed as the junction of a pair of co-ordinate lines: ray and circle; each pair of such co-ordinate lines is fixed as having such a junction-point. Each ray is known and named from its



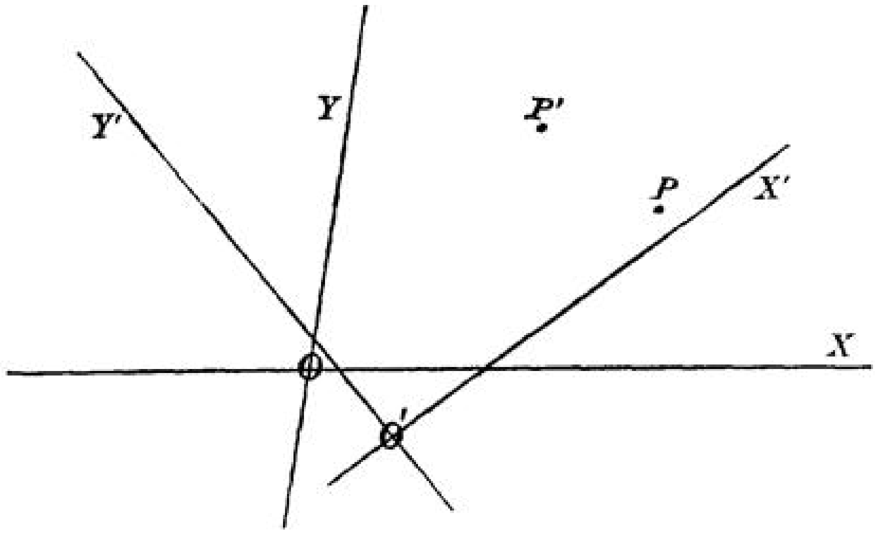
angle with some assumed fixed ray, as OD, called **base-line** or **polar-axis**; angles reckoned *clockwise*, as θ', are marked $-$, those reckoned *counter-clockwise*, as θ, are marked $+$; the direction of a ray which *bounds the ray's angle*, as OP, is taken $+$; the *counter*-direction, as OP'', $-$. Each circle is known and named from the radius ρ. The angle θ of a ray and the radius ρ of a circle are called the *polar co-ordinates* of the junction-point of ray and circle; i.e., of the point (ρ, θ). $\rho = c$ is the Eq. of a circle: it declares that every point of the circle is distant c from O; $\theta = d$ is the Eq. of a ray: it declares that the radius vector from O to any point of the ray is sloped d to the polar axis. Ray and circle meet in the point whose Eqs. are $\rho = c$, $\theta = d$, i.e., in the point (ρ, θ).

N.B. To fix all points in the plane by rectilinear co-ordinates, x and y, it is necessary to let each range in value from $-\infty$ to $+\infty$; but it is sufficient to let ρ range from 0 to $+\infty$. θ from 0 to 2π. So confining ρ and θ we have but one pair of values to fix any one point; but if we let each range from $-\infty$ to $+\infty$, then any point having co-ordinates (ρ, θ) will also have co-ordinates $(-\rho, \theta + \pi)$, $(-\rho, -\pi + \theta)$, $(\rho, -2\pi + \theta)$, and in each of these four pairs we may suppose θ increased or decreased by $2n\pi$, n being any natural number. In polar co-ordinates with this range, to any pair of values corresponds but one point, but to any point correspond *four infinities* of pairs of values.

11. By reasoning quite like that of Art. 7 it may now be shown that the geometric picture of any Eq. between polar co-ordinates, as $\phi(\rho, \theta) = 0$, is a plane curve. Every point of the curve pictures a pair of values of ρ and θ satisfying the Eq. $\phi(\rho, \theta) = 0$; conversely, every pair of values of ρ and θ satisfying the Eq. $\phi(\rho, \theta) = 0$ is pictured by a point of the curve. We speak indifferently of the Eq. $\phi(\rho, \theta) = 0$ and of the curve $\phi(\rho, \theta) = 0$.

12. There are various other kinds of co-ordinates, as bipolar, trilinear, homogeneous, elliptic; but rectilinear (called also Cartesian, from Descartes, the inventor) and polar are the most common and important.

A point that may be anywhere in a plane and a pair of co-ordinates that may have any values may be said to have *two* degrees of freedom; a point that may be anywhere on some curve in a plane and a pair of co-ordinates whose values must satisfy some Eq. may be said to have *one* degree of freedom; a point that must have one of several definite positions and a pair of co-ordinates that must have one of several definite sets of values may be said to have *no* degree of freedom. Thus it is seen that mobility in the point corresponds to variability in the pair of values.



It is to be noted that the same pair of values will in general be pictured by different points, not only in systems of different co-ordinates, as rectilinear and polar, but also in different systems of the same co-ordinates. Thus the pair (2, 1) is pictured by the point P in the system OX, OY, but by the point P' in the system $O'X'$, $O'Y'$. Conversely, in different co-ordinate systems the same point will picture different pairs of values. See Art. 21.

EXERCISES.

Assume a system of rectangular axes, also take the $+X$-axis as a polar axis; then,

1. Find the point $\left(\rho = 2, \theta = \frac{\pi}{6}\right)$, and show that its rectangular co-ords. are $(\sqrt{3}, 1)$.

2. Show that the rectang. co-ords. of (ρ, θ) are $x = \rho \cos\theta$, $y = \rho \sin\theta$.

3. Hence, express ρ and θ through x and y.

4. Find the rectang. and polar Eqs. of the axes and of a circle about O, radius 5.

Summary.

13. The results reached so far may thus be summed:

A. *A pair of numbers* (as corresponding values of argument and function) *may be pictured geometrically by a point in a plane.*

A′. *A point in a plane may be pictured algebraically by a pair of numbers* (the co-ordinates of the point).

The point picturing a number-pair and the number-pair picturing a point will vary with the co-ordinate system chosen.

B. *An equation* (or functional relation) *between two symbols* (as $f(x, y) = 0$, $\phi(\rho, \theta)$, *may be pictured geometrically by a plane curve.*

B′. *A plane curve may be pictured algebraically by an equation between two symbols* (called *current co-ordinates* of a point of the curve).

The curve picturing an equation and the equation picturing a curve will vary with the co-ordinate system chosen.

Strict proof of B′ is neither in place nor needed here; as occasion may offer it will be verified as we go on.

14. The doctrine based on these facts is named **Co-ordinate** (or Analytic, or Algebraic) **Geometry** *of the Plane.*

Its problem is twofold:

I. *Given any algebraic form* (as $f(x, y) = 0$), *to picture it by a geometric form in a plane* (a plane curve) *and to interpret its properties as geometric properties* (of the curve).

II. *Given any geometric form in a plane* (a plane curve), *to picture it by an algebraic form* (as $f(x, y) = 0$) *and thence deduce its properties algebraically.*

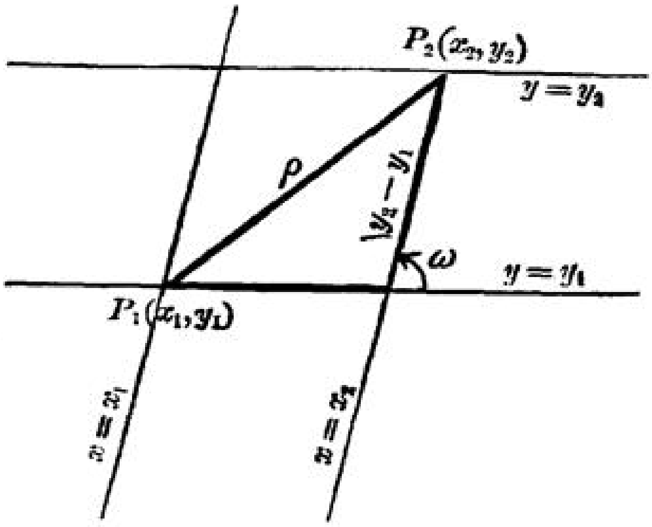
N.B. Not to repel by over-subtlety, by a number has thus far been meant a (so-called) *real* number. But, as the student may know, there are equations whose roots (some or all) are (so-called) *imaginary* numbers, i.e., numbers involving the symbol i or $\sqrt{-1}$; as $x^2+y^2=-1$. This Eq. is satisfied by no pair of *real* values of x and y; hence it *cannot* be pictured geometrically in the plane of the axes OX, OY, in which, e.g., $x^2+y^2=1$ is pictured by a circle about O with radius 1, since every point of this plane pictures and pictures only a pair of real values of x and y. If, then, $x^2+y^2=-1$ can be pictured geometrically at all, it must be by some geometric form *not* in the plane of OX, OY. The whole question of the depiction of pairs of imaginary numbers must be reserved.

A *real* number may be defined as one whose *second* power is *positive*; an *imaginary*, as one whose *second* power is *negative*; a *complex* number, as made up of a *real* and an *imaginary* part.

CHAPTER II.

THE RIGHT LINE.

Before attacking the problem proper of Co-ordinate Geometry it may be well to establish certain useful elementary relations.



15. Distance between two points in terms of their rectilinear co-ordinates. Let the points P_1, P_2 picture the pairs written beside them. Then, at once,

$$d^2 = \overline{x_2 - x_1}^2 + \overline{y_2 - y_1}^2 - 2\,\overline{x_2 - x_1}\cdot\overline{y_2 - y_1}\cdot\cos(\pi - \omega),$$

or, $$d^2 = \overline{x_2 - x_1}^2 + \overline{y_2 - y_1}^2 + 2\,\overline{x_2 - x_1}\cdot\overline{y_2 - y_1}\cdot\cos\omega.$$

N.B. We may as well write $x_1 - x_2$ *and* $y_1 - y_2$.

Corollary 1. For $\omega = 90°$, $\cos\omega = 0$;

$$\therefore d^2 = \overline{x_2 - x_1}^2 + \overline{y_2 - y_1}^2; \quad \text{i.e.,}$$

The squared distance between two points equals the sum of the squared rectangular co-ordinate differences of the points.

Cor. 2. If one of the points, as P_1, be the origin, then

$$x_1 = 0, \quad y_1 = 0; \quad \therefore d^2 = x_2^2 + y_2^2 + 2\,x_2 y_2 \cos\omega; \quad \text{i.e.,}$$

The squared distance of a point from the origin equals the sum of its squared rectilinear co-ordinate differences plus twice their product by the cosine of the co-ordinate angle.

N.B. It will be convenient to use certain self-explaining symbols and abbreviations: as, $\perp$ for perpendicular; $\triangle$ for triangle; $\measuredangle$ for angle; ∟ for right angle; $\parallel$ for parallel; Eq. for equation; Cd. for co-ordinate; RL. for right line; $P(x, y)$ or simply P, or simply (x, y) for the point whose co-ordinates are x and y; P_1 or (x_1, y_1) for the point whose co-ordinates are x_1 and y_1; and so for other subscripts, *the uniform use of the subscript being to limit a general symbol;* also $\overline{P_1P_2}$ or $\overline{(x_1, y_2)\,(x_2, y_2)}$ for the tract from $P_1\,(x_1, y_1)$ to $P_2\,(x_2, y_2)$.

EXERCISES.

1. If (ρ_1, θ_1), (ρ_2, θ_2) be two points, δ their distance apart, show that $\delta^2 = \rho_1{}^2 + \rho_2{}^2 - 2\rho_1\rho_2 \cos \overline{\theta_1 - \theta_2}$.

2. The vertices of a $\triangle$ are $(2, 4)$, $(-2, 7)$, $(-6, -8)$; draw it, and find lengths of its sides, for $\omega = 90°$, and for $\omega = 60°$.

3. Draw the 4-side (quadrilateral) whose vertices are $(7, 2)$, $(0, 9)$, $(-3, -1)$, $(-6, 4)$, and find lengths of sides and diagonals, for $\omega = 90°$, and $\omega = 45°$.

4. Find length of tract between $(17, 30° \, 11')$ and $(19, 48° \, 26')$.

5. Find points on Y-axis distant d from (x_1, y_1), for $\omega = 90°$.

6. Say (by an Eq.) that (x_1, y_1) is distant 11 from $(7, -2)$; $\omega = 60°$.

7. Say that (x, y) is equidistant from $(2, 5)$ and $(-11, 1)$; $\omega = 45°$.

8. Find (x, y) equidistant from $(2, -13)$, $(-9, 5)$, $(17, 23)$, $\omega = 90°$.

9. The tract $\overline{(5, -3)(22, y)}$ is $\sqrt{314}$ long; find y, for $\omega = 90°$.

10. When is $(4, 5)$ equidistant from $(-3, 1)$ and $(9, -2)$?

16. If the Cds. (x', y') satisfy an Eq. $f_1(x, y) = 0$, then is the point (x', y') on the curve $f_1(x, y) = 0$; if the same pair satisfy a second Eq. $f_2(x, y) = 0$, then the same point (x', y') is on a second curve $f_2(x, y) = 0$; conversely, if (x', y') be a common, or junction, point of two curves: $f_1(x, y) = 0$, $f_2(x, y) = 0$, then the pair (x', y') satisfies both Eqs., i.e.,

$f_1(x', y') = 0$ and $f_2(x', y') = 0$. Hence, *to find the Cds. of the junction-points of two curves,* **solve their Eqs. as simultaneous.** From Algebra we know that the solution of two simultaneous Eqs., one of pth and one of qth degree, involves in general the solution of one Eq. of pqth degree; such an Eq. has pq roots; therefore, there will be in general pq pairs of values of x and y satisfying both Eqs.; i.e., two curves, one of pth and one of qth degree, meet in pq points. Two, three, or many pairs of values may be equal, in which case there is a double, triple, or multiple common point; two or $2n$ pairs may be imaginary, where two or $2n$ common points are imaginary, not in our plane of OX, OY.

ILLUSTRATIONS. **1.** $3x - 7y = 55$ and $5x + 2y = -4$ meet in $(2, -7)$.

2. $x^2 + (5 - y)(5 - \frac{5}{2}x - y) = 0$ and $x + y = 7$ meet in (4.3) and $(-2, 9)$.

3. $9x^2 + 10xy + y^2 = 273$ meets $9x^2 - 10xy + y^2 = 33$ in $(1, 12)$, $(-1, -12)$, $(4, 3)$, $(-4, -3)$.

4. $y^2 = 4x$, $x^2 = 4y$; hence, $y^4 - 64y = 0$; or $y(y-4)(y^2 + 4y + 16) = 0$; common are $(0, 0)$, $(4, 4)$, $(-2 - i2\sqrt{3}, -2 + i2\sqrt{3})$, $(-2 + i2\sqrt{3}, -2 - i2\sqrt{3})$.

The last two pairs cannot be pictured by points in our plane; in what sense they are section-points of the two curves in our plane cannot now be made clear.

On review let the student construct the above curves.

17. Cds. of the point that divides a tract $\overline{P_1P_2}$ in a given ratio $\mu_1 : \mu_2$.

If $P(x, y)$ be the point, so that

$$P_1P : PP_2 = \mu_1 : \mu_2,$$

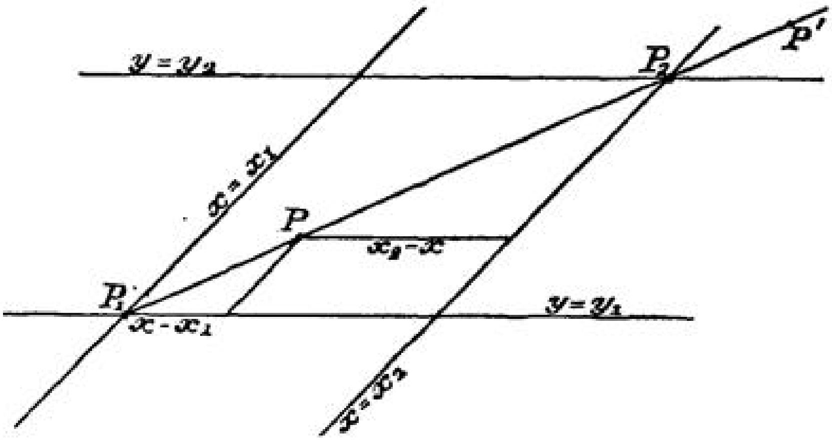
then, by similar $\triangle$, $x - x_1 : x_2 - x = \mu_1 : \mu_2$;

$$\text{or,} \qquad x = \frac{\mu_1 x_2 + \mu_2 x_1}{\mu_1 + \mu_2};$$

likewise, $y = \dfrac{\mu_1 y_2 + \mu_2 y_1}{\mu_1 + \mu_2}$. Notice the order of the subscripts.

If either term of the *ratio*, as μ_2, be *negative*, then is $P' P_2$ to reckon *counter* to $P' P_1$; i.e., P' falls *without* the tract, next to P_2. The division is then called *outer*. Conversely, if the division be *outer*, one term of the ratio, and hence the *ratio* itself, is *negative*.

The formulæ yield only one pair of values of x and y; hence, only one point divides a tract in a given ratio.



$P_1 P$ and $P P_2$ (or $P_1 P'$ and $P' P_2$) are called **segments** of the tract $P_1 P_2$, and $P_1 P : P P_2$ (or $P_1 P' : P' P_2$) is called the **distance-ratio** of P (or P') to P_1 and P_2. This *distance-ratio* is + or − according as the division is *inner* or *outer*.

If P be the (inner) mid-point of the tract, then $\mu_1 = \mu_2$,

$$\therefore x = \overline{x_1 + x_2} : 2, \quad y = \overline{y_1 + y_2} : 2; \quad \text{i.e.,}$$

the Cds. of the (inner) *mid-point of a tract are the half-sums of the like Cds. of its ends.*

If $\mu_1 = -\mu_2$, P' is the outer mid-point, and x and y are *infinite; the outer mid-point of a tract is at* ∞* on the RL. the tract is part of.

EXERCISES.

1. Find the Cds. of the points which divide the tract $(7, 11)(3, 5)$ in the ratio 2 : 3, and the tract $(3, 13)(-7, -1)$ in the ratio 3 : −4.

* See note, page 196.

2. The vertices of a △ are (x_1, y_1), (x_2, y_2), (x_3, y_3); find the points that cut its medials in the ratio 2 : 1, reckoning from the vertices.

HINT. Take care to think as much, and reckon as little, as possible. Here, taking any vertex, we find the x of the division-point on the medial is $\overline{x_1 + x_2 + x_3} : 3$; ∴ the y is $\overline{y_1 + y_2 + y_3} : 3$. These expressions, being symmetric, like-formed, as to the subscripts, hold for all the medials; ∴ the 3 points fall together, are one. This point is named *mass-centre* of the △.

3. A point P starts from (x_1, y_1) and moves *half*-way toward (x_2, y_2), then turns and moves one-*third* of the way toward (x_3, y_3), then one-*fourth* of the way toward (x_4, y_4), and so on, till at last it moves one nth of the way toward (x_n, y_n); where does it stop?

The final position of P is called *mid-centre* or *mean point* of the n points.

4. The point P starts from P_1 and moves over $\frac{\mu_2}{\mu_1}$ of the way toward P_2, then over $\frac{\mu_3}{\mu_1 + \mu_2}$ of the way toward P_3, then over $\frac{\mu_4}{\mu_1 + \mu_2 + \mu_3}$ of the way toward P_4, and so on; where does it stop?

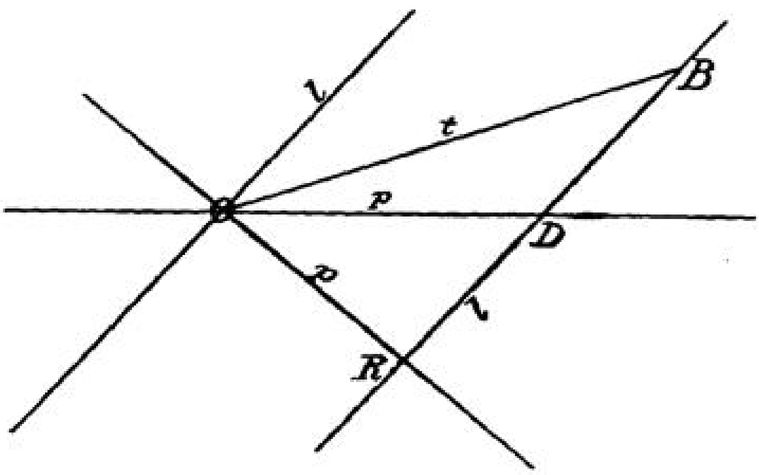
The final position of P is called *centre of proportional distances.*

5. Three vertices of a parallelogram are (x_1, y_1), (x_2, y_2), (x_3, y_3); find the Cds. of the 4th and of mid-points of the diagonals.

What do the results mean geometrically?

Parallel Projections.

18. *The intercept, on any RL., made by two ∥ planes through the ends of a tract is named* **parallel projection** *of the tract on the RL.* If the planes be ⊥ to the RL., the projection is



orthogonal; otherwise, *oblique*. Thus, OR and OD are projections of OB: OR, orthogonal; OD, oblique.

Clearly, projections of the same tract by the same planes on ‖ RLs. are equal. Accordingly, in comparing the lengths of a tract and its projections, we may suppose all the lines of projection to pass through one end of the tract. Calling the tract t, its projection p, and denoting by $\widehat{dd'}$ the angle from any direction d reckoned around to any other direction d', we have by the Law of Sines,

$$p : t = \sin \widehat{tl} : \sin \widehat{pl}, \quad \text{whence} \quad p = t \cdot \frac{\sin \widehat{tl}}{\sin \widehat{pl}},$$

i.e., *the projection of a tract equals the product of the tract and the quotient of the sine of the slope of the tract to the direction of projection by the sine of the slope of the projection to the same direction.*

By odds the most important ‖ projection is the *orthogonal.* The ∡ of a tract with a line of orth. proj. is named *direction-angle*, its cosine is the **direction-cosine** of the tract. Since the ∡ of a RL. with a plane is the complement of its ∡ with a ⊥ to the plane, we have $p = t \cdot \cos \widehat{pt}$; i.e., *orth. proj. of a tract = product of the tract by its direction-cosine.*

19. If we project on any RL. the sides of any closed polygon, taken in order, the *end* of the projection of the *last* side will fall on the *beginning* of the projection of the *first;* i.e.,

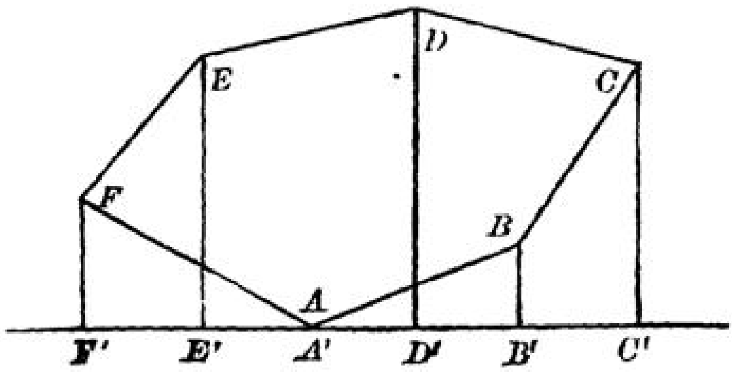
The sum of the projections of the sides of a closed polygon is 0.

Hence, *The projection of any side equals the negative sum of the projections of the other sides.*

Or, *The sum of the projections of a train of tracts between two points equals the projection of the one tract between them.*

If the tracts projected and the RLs. they are projected on lie all in one plane, we may put projecting RLs. for projecting planes. For this, the figure illustrates the above propositions.

We see $A'B' : AB = \sin A'BB' : \sin AB'B$. If $AB'B = 90°$, $A'B' = AB \cdot \cos BA'B'$. $A'B'$ is the projection both of AB and of $AFEDCB$; the projection of $BCDEFA$ is $B'A'$ or $-A'B'$.



20. The tract from the origin to any point is called the *radius vector* of that point. In the light of the above we may now define:

The rectilinear cds. of a point are the **projections** *of its* **radius vector** *on each of two axes in its plane,* ‖ *to the other axis.*

The *polar Cds.* of a point are its *radius vector* and the *direction-angle* of its radius vector reckoned from the polar axis.

The doctrine of projections is of prime importance in mathematics. It is here used to treat the

Transformation of Co-ordinates.

21. The Cds. of a point vary with the system of Cds. (Art. 12). To express the Cds. of a point, in one system, through the Cds. of the same point, in another system, is to *transform* the Cds.

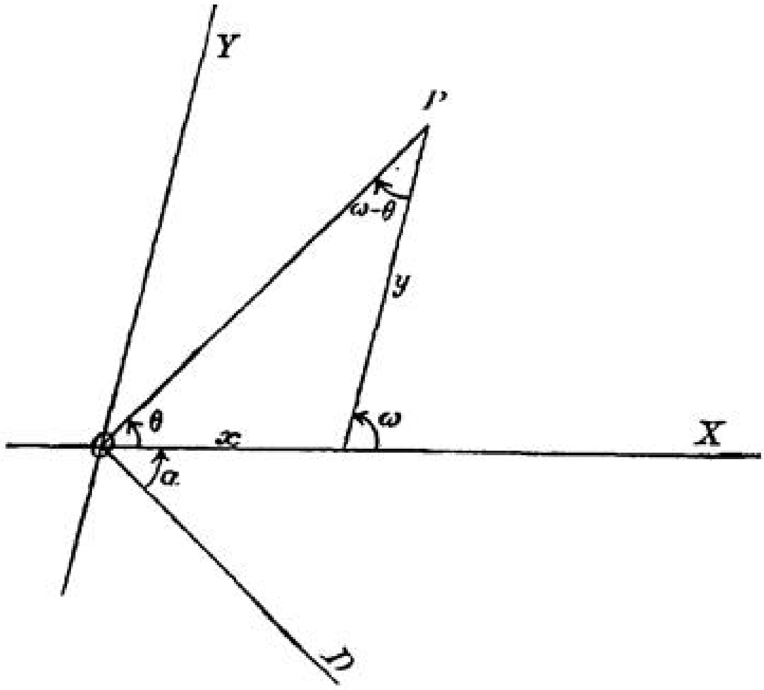
Several cases arise.

I. *To pass from rectilinear to polar Cds.*, the origin being the same for both. From the above definitions, or from the figure, we have at once:

$$x : \rho = \sin \widehat{\rho y} : \sin \widehat{xy} = \sin(\omega - \theta) : \sin \omega,$$

$$y : \rho = \sin \widehat{x\rho} : \sin \widehat{xy} = \sin \theta : \sin \omega;$$

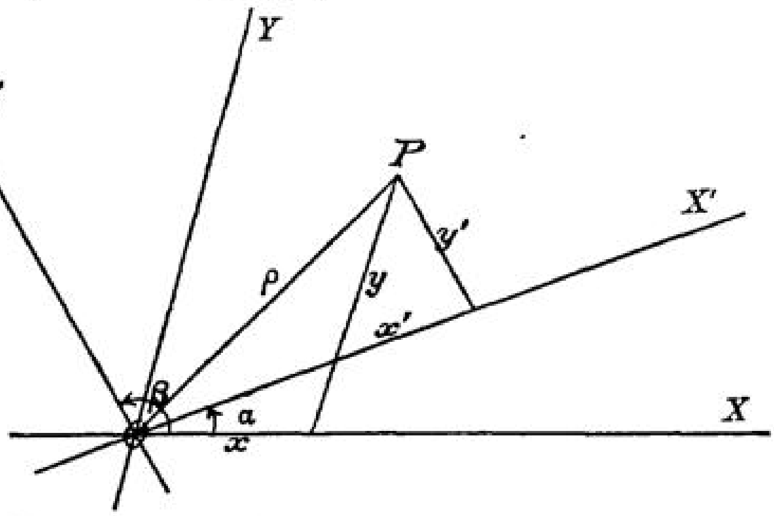
$$\therefore x = \rho \cdot \frac{\sin(\omega - \theta)}{\sin \omega}, \qquad y = \rho \cdot \frac{\sin \theta}{\sin \omega}.$$



These Eqs. presume that the X-axis is the polar axis; if the X-axis be sloped α to the polar axis, put $\theta - \alpha$ for θ.

For $\omega = 90°$, $\quad x = \rho \cos \theta, \quad y = \rho \sin \theta.$

II. *To pass from one rectilinear system to another with same origin.* By Art. 19 the proj. of ρ, on $OX \parallel$ to OY, equals the sum of the projs. of x' and y';



$$\therefore x = x' \cdot \frac{\sin \widehat{x'y}}{\sin \widehat{xy}} + y' \cdot \frac{\sin \widehat{y'y}}{\sin \widehat{xy}}; \text{ or, } x \cdot \sin \widehat{xy} = x' \sin \widehat{x'y} + y' \sin \widehat{y'y},$$

and $$y \cdot \sin \widehat{xy} = x' \sin \widehat{xx'} + y' \sin \widehat{xy'}.$$

This last Eq. is got by first exchanging x and y in the Eq. above it, which amounts to projecting on $OY \parallel$ to OX, and then exchanging the letters in the angles, it being remembered that $\widehat{ab} = -\widehat{ba}$ and $\sin \widehat{ab} = -\sin \widehat{ba}$.

Angles are best reckoned *from* the $+X$-axis or *toward* the $+Y$-axis.

If the X'- resp. Y'-axis be sloped α resp. β to the X-axis, we may write

$$x \sin \omega = x' \sin(\omega - \alpha) + y' \sin(\omega - \beta),$$

$$y \sin \omega = x' \sin \alpha + y' \sin \beta.$$

From these general formulæ the student may find special ones:

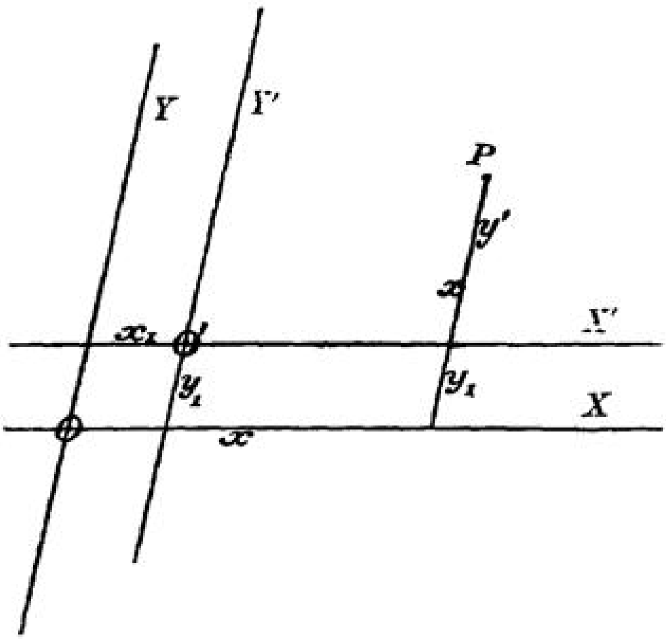
(*a*) For passing from rectangular to oblique axes.

(*b*) For passing from oblique to rectangular axes.

(*c*) For passing from rectangular to rectangular axes.

The results are not so symmetric and easy to recall as the general formulæ. Let the student draw the figures and interpret geometrically each term in each Eq.

III. *To pass to parallel axes through a new origin.* Be OX, OY the old axes, $O'X'$, $O'Y'$ the new ones; x', y', the old Cds. of the new origin O'. If x, y resp. x', y' be the old resp. new Cds. of P, we have



$$x = x' + x_1, \quad y = y' + y_1,$$

i.e., *for the old Cds. put the new Cds. plus the old Cds. of the new origin.*

If we will change both origin and axial directions, we can change first either, then the other, or both at once, by adding to the expressions for the old Cds. the old Cds. of the new origin. Calling these latter, as above, x_1, y_1, and putting q_1, q_2, q_1', q_2'

for the sine-quotients in II., we get as the most general relations between the Cds. of a point referred to two systems:

$$x = x_1 + q_1 x' + q_2 y', \qquad y = y_1 + q_1' x' + q_2' y.$$

Conversely, such a pair of Eqs. may always be interpreted as a transformation of Cds. For x_1, y_1 may be taken as old Cds. of a new origin (or negative new Cds. of an old origin), and we can find ω, α, and β such that $\sin(\omega - \alpha) : \sin\omega = q_1$, $\sin(\omega - \beta) : \sin\omega = q_2$, etc.

22. Note that the general Eqs. of transformation are homogeneous of 1st degree in Cds. So much might have been assumed, it being clear that a length, as x or y, can be expressed only as made up of lengths. Under this assumption, by determining the values of x_1, y_1, and the q's, the student may now get the Eqs. already found; this is recommended as a useful exercise.

The magnitudes x_1, y_1, and the q's, are *not* of the same class; the latter are *pure* numbers, trigonometric ratios, while the former are Cds., *metric* numbers of tracts. A number is said to have as many dimensions as the geometric magnitude it stands for: the metric number of a *length*, *area*, resp. *volume* has *one*, *two* resp. *three* dimensions. A pure number, the ratio of two like metric numbers, is of 0th degree or dimension. Thus, $64^1 \equiv 8^2 \equiv 4^3$ has one, two, three dimensions, according as it is the metric number of a length, an area, a volume.

It is plain that any Eq. may be thought as homogeneous by thinking the numeral coefficients of proper dimensions.

23. If in any Eq. $f(x, y) = 0$, we put for x and y any **linear** functions (i.e., functions of 1st degree) of x' and y', we are said to make a **linear** *substitution* or *transformation*. Such a substitution may, of course, change the form of the Eq., but *it will* not *change its degree in x and y*. For it cannot *raise* the degree, since any term or factor of a term, as x^r, will be replaced by a series of terms, none of degree higher than the

rth, in x' and y'; neither can it lower it, since then by expressing x' and y' as clearly as we can, *linearly* through x and y, and re-substituting, we should get the original Eq., and so *raise* the degree by a *linear* substitution, which is impossible.

This again we might have foreseen. For the picture of $f(x, y) \doteq 0$ is a curve whose degree tells the number of points in which it may be cut by a RL. (see Arts. 24, 16); a linear substitution is interpreted geometrically as a change of axes; by such a change we in no wise affect the curve, hence do not change the number of points in which a RL. cuts it; hence we do not change the degree of the Eq.

The doctrine of Transformation of Cds. is of special importance to Mechanics. Any motion of a plane system of points may be resolved into a *push* and a *turn*. A *push* corresponds to a change of *origin* simply, a turn to a change of *axial directions;* a *twist* corresponds to a change of *both*.

ILLUSTRATIONS. 1. Transform $x^2 + 14x + y^2 - 10y + 49 = 0$ to ‖ axes through $(-7, 5)$.

We have

$$x = x' - 7, \quad y = y' + 5;$$

$$\therefore (x'-7)^2 + 14(x'-7) + (y+5)^2 - 10(y'+5) + 49 = 0;$$

or, $$x'^2 + y'^2 = 25.$$

2. $x^2 - y^2 = a^2$. Pass to axes halving the ∡s between the old ones.

We have

$$\omega = 90°, \quad \alpha = 45°, \quad \beta = 135°;$$

$$\therefore \omega - \alpha = 45°, \quad \omega - \beta = -45°;$$

$$\therefore x = x' \sin 45° - y' \sin 45° = \frac{x' - y'}{\sqrt{2}};$$

$$y = x' \sin 45° + y' \sin 135° = \frac{x' + y'}{\sqrt{2}};$$

whence, substituting and dropping accents,

$$2xy = -a^2.$$

3. $3x^2+4xy+5y^2-5x-7y-21=0=f(x,y)$. Change the origin and turn the axes, keeping them rectangular, so as to make the terms containing the first powers and the product of x and y vanish.

Putting $x'+x_1$ for x, $y'+y_1$ for y, and collecting, we get

$$3x'^2+4x'y'+5y'^2+2(3x_1+2y_1-\tfrac{5}{2})x'$$
$$+2(2x_1+5y_1-\tfrac{7}{2})y'+f(x_1,y_1)=0.$$

If the terms in x' and y' vanish, then

$$3x_1+2y_1-\tfrac{5}{2}=0, \qquad 2x_1+5y_1-\tfrac{7}{2}=0;$$

whence, $\quad x_1=1:2, \quad y_1=1:2.$

Hence, $f(x_1,y_1)=f(\tfrac{1}{2},\tfrac{1}{2})=3\cdot\tfrac{1}{4}+4\cdot\tfrac{1}{4}+5\cdot\tfrac{1}{4}-7\cdot\tfrac{1}{2}$
$$-5\cdot\tfrac{1}{2}-21=-24.$$

Accordingly, on passing to new ∥ axes, through $(\tfrac{1}{2},\tfrac{1}{2})$, the Eq. becomes

$$3x'^2+4x'y'+5y'^2-24=0.$$

Now turn the axes through an ∡ α; then, X and Y being new axes,

$$x'=x\cos\alpha-y\sin\alpha, \qquad y'=x\sin\alpha+y\cos\alpha;$$

$$\therefore\ (3\,\overline{\cos\alpha}^2+5\,\overline{\sin\alpha}^2+4\sin\alpha\cdot\cos\alpha)x^2$$
$$+(3\,\overline{\sin\alpha}^2+5\,\overline{\cos\alpha}^2-4\sin\alpha\cdot\cos\alpha)y^2$$
$$+(4\sin\alpha\cdot\cos\alpha+4\,\overline{\cos\alpha}^2-4\,\overline{\sin\alpha}^2)xy=24.$$

If the term in xy vanishes, then

$$\sin\alpha\cdot\cos\alpha+\overline{\cos\alpha}^2-\overline{\sin\alpha}^2=0,$$

or, $\quad \tfrac{1}{2}\sin 2\alpha=-\cos 2\alpha, \quad$ or, $\quad \tan 2\alpha=-2.$

Hence, on reduction,

$$(4+\sqrt{5})x^2+(4-\sqrt{5})y^2=24,$$

α is 58° 16′ 57″ or $-\{$31° 43′ 3″$\}$; but it is needless to find α from the tables; it is much better to construct it geometrically.

We pass now to the geometric interpretation of Eqs., and naturally begin with the general

Equation of First Degree in x and y.

24. This has the form $lx + my + n = 0$.

To interpret it, let us pass to a new system of Cds. x' and y', such that $x' = lx + my + n$. Every point whose *old* Cds., x and y, satisfy $lx + my + n = 0$ has its *new* Cd. $x' = 0$, and clearly all such points are on the Y'-axis; again, every point on the Y'-axis has its $x' = 0$, and hence has such *old* Cds., x and y, as satisfy $lx + my + n = 0$; this Y'-axis is a RL.; hence *every point whose rectilinear Cds., x and y, satisfy an Eq. of 1st degree in x and y lies on a certain RL., and the Cds., x and y, of every point on that RL. satisfy that Eq.*

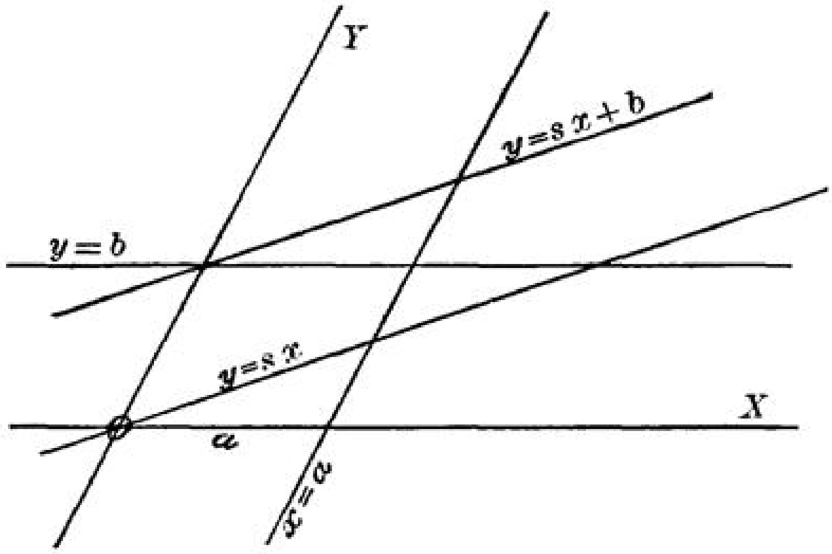
Conversely, suppose given any RL., and seek the form of its Eq. Assume it as a new Y'-axis; for all points on it and for no others $x' = 0$; but any Cd., x', is a linear function of the old Cds., x and y; i.e., $x' = lx + my + n$; hence for all points on this RL. and for no others $lx + my + n = 0$. Therefore,

I. *The geometric picture of any Eq. of 1st degree in rectilinear Cds., x and y, is a RL.*

II. *The algebraic picture of any RL. is an Eq. of 1st degree in rectilinear Cds., x and y.*

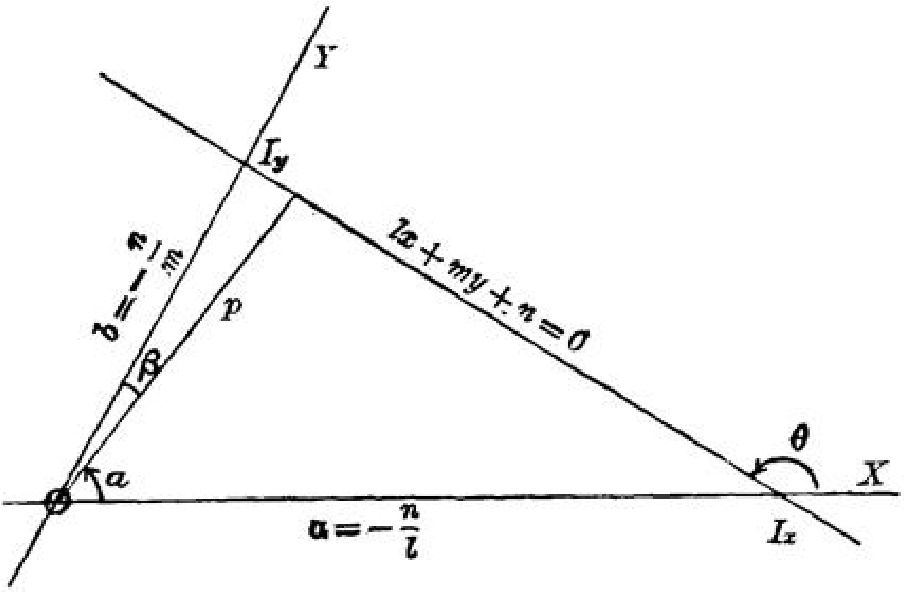
We speak indifferently of the Eq. and of the RL.: $lx + my + n = 0$. A convenient abbreviation for $lx + my + n$ is L; if l, m, n, have any subscript, L has the same subscript, and conversely, so that $L_k \equiv l_k x + m_k y + n_k$. We shall also speak of the right line L, meaning the RL. whose Eq. is $L = 0$.

The above proof is the most natural, and presents no difficulty; but owing to the great importance of the proposition, it will be well for the student to frame a proof from the figure: showing that the Eqs. of the 4 RLs. are really such as are written by them; thus showing that by changing a, b, s one of these Eqs. may be made to fit any RL. that may be drawn in the plane.



25. The values of x and y range in pairs picturing the points of the RL. each from $-\infty$ to $+\infty$; hence x and y are called *running* or *current* Cds.—For any one RL. l, m, n are not definite, since we may multiply the Eq. by any expression we will, without changing the relation between x and y. But the *ratios* $l:m$, $m:n$, $n:l$ are *fixed* for any *one* RL., *different* for *different* RLs. They are called *arbitraries* or *parameters*. Their number is apparently *three*, really *two*, for the *third* is but the quotient of the other two. Clearly they are not changed by multiplying the Eq. at will. To interpret them, assume any axes, and construct the RL. To do this it suffices to know *two* points of the RL. or *one* point and the *direction*. To find a point, we must find a *pair* of values of x and y satisfying the Eq. To do this, we may assign any value, say, to x, and reckon the corresponding value of y. The simplest value we can assign to x is 0; the corresponding point will be on the Y-axis, since only on the Y-axis are points whose x is 0. The corresponding value of y is $-n:m$. This, then, is the distance OI_y from the origin at which the RL. cuts the Y-axis. Likewise, putting $y=0$, we find the distance from the origin at which the RL. cuts the X-axis to be $-n:l$. The RL. making these intercepts, $-n:m$ resp. $-n:l$ on the Y- resp. X-axis, is the RL. $lx+my+n=0$.

Two of the ratios are thus seen to represent negative intercepts made on the axes by the RL. Denote these intercepts on the X- resp. Y-axis by a resp. b, so that $a\equiv -n:l$, $b\equiv -n:m$. Then, on transposing n and dividing by it, the Eq. becomes $\frac{x}{a}+\frac{y}{b}=1$, which is the *Intercept Form* (I.F.).



The 3d ratio $-l:m\equiv -\frac{b}{a}$ let us denote by the symbol s and name the Direction-Coefficient of the RL. Note that s is the negative ratio of the coefficients of x and y. If θ be the slope of the RL. to the X-axis, then clearly

$$s=\sin\theta:\sin(\omega-\theta).$$

Hence for $\omega=90°$, i.e., for rectang. axes, $\boldsymbol{s=\tan\theta}$.

Solving the Intercept Form as to y, we get $\boldsymbol{y=sx+b}$, which is the *Directional Form* (D.F.).

Drop from O on $lx+my+n=0$, a $\perp p$ sloped α resp. β to the X- resp. Y-axis; then $p=a\cos\alpha=b\cos\beta$.

Substituting in the I.F. for a and b, we get

$$\boldsymbol{x\cos\alpha+y\cos\beta-p=0,}$$

which is the *Normal Form* (N.F.).

For $\omega = 90°$, $\cos\beta = \sin\alpha$; hence, we get

$$\boldsymbol{x \cos \alpha + y \sin \alpha - p = 0},$$

an important special form.

Hence we have the following Rules:

To bring the Eq. of a RL. to the I.F., divide by the absolute term taken to the right member. This is possible unless the absolute be 0; then the pair (0, 0) satisfies the Eq., and the RL. goes through the origin. — In general, *if the absolute in any Eq., of any degree, between Cds. be 0, the curve goes through the origin.* — To construct a RL. through the origin, assign either Cd. any convenient value, then reckon the other Cd.; this pair pictures a point, which with the origin fixes the RL.

To bring to the D.F., solve as to y. This is possible unless the coefficient of y be 0, i.e., unless y does not appear at all; then the Eq. reduces to x = a constant, — the RL. is ∥ to the Y-axis. Likewise, if x does not appear, y = a constant, — the RL. is ∥ to the X-axis.

To bring to the N.F., multiply by the normalizing factor F.

To find F, we note that, since by hypothesis

$$Flx + Fmy + Fn \equiv x\cos\alpha + y\cos\beta - p,$$

$$Fl = \cos\alpha, \quad Fm = \cos\beta, \quad Fn = -p,$$

whence, $F = -p : n$.

$$p\sqrt{\left\{\frac{n^2}{l^2} + \frac{n^2}{m^2} - 2\frac{n}{l}\cdot\frac{n}{m}\cos\omega\right\}} = \frac{n}{l}\cdot\frac{n}{m}\sin\omega,$$

since each is the double area of the $\triangle\ OI_xI_y$. Hence

$$\boldsymbol{F = \sin\omega : \sqrt{l^2 + m^2 - 2lm\cos\omega}}.$$

Before the $\sqrt{\ }$ we have choice of signs; we agree to take always that sign which will make the absolute negative.

For $\omega = 90°$, the most important case,

$$\boldsymbol{F = 1 : \sqrt{l^2 + m^2}}.$$

EXERCISES.

Construct the following RLs., reduce the Eqs. to the various forms, for $\omega = 90^\circ$, unless otherwise stated:

1. $5x - 3y + 30 = 0.$

The I.F. is $\dfrac{x}{-6} + \dfrac{y}{10} = 1$, got by dividing by -30;

the D.F. is $y = \frac{5}{3}x + 10$, got by solving as to y;

the N.F. is $-\dfrac{5}{\sqrt{34}}\,x + \dfrac{3}{\sqrt{34}} = 0$, got by multiplying by $F = -1 : \sqrt{34}$.

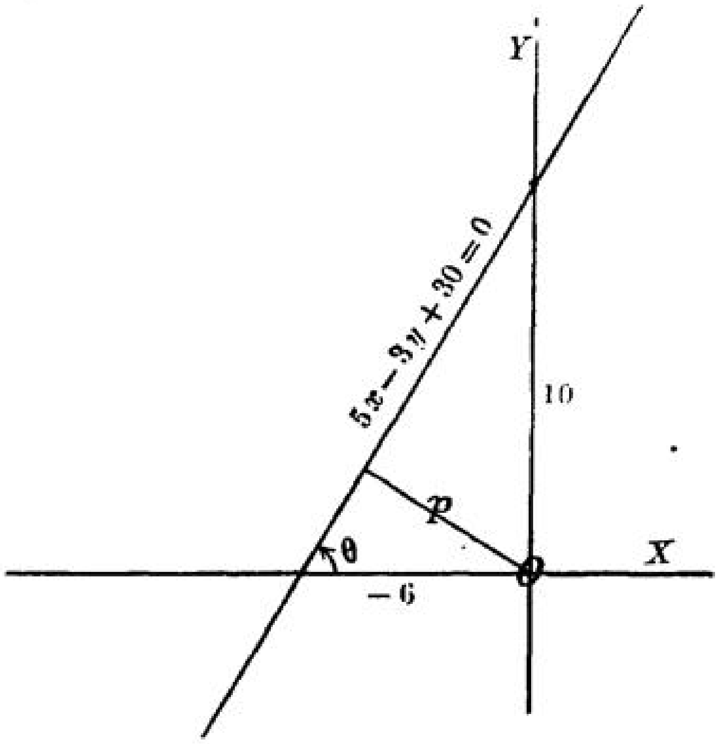
$a = -6,\quad b = 10,\quad s = 5 : 3 = \tan\theta,\quad p = 30 : \sqrt{34}.$

For $\omega = 60^\circ$, a, b, s are the same, but $F = -\sqrt{3} : 14$, $p = 15\sqrt{3} : 7$.

2. $3x + 7 = 0.$ 4. $3x - 2y = 0.$ 6. $2x + 4y = 9$ ($\omega = 60^\circ$).

3. $7y - 9 = 0.$ 5. $3x - 4y = 12.$

7. How are $x + y = 0$ and $x - y = 0$, $x - y + a = 0$ and $x + y + b = 0$ related (ω at will)?



26. *Angle between* **two RLs.**: $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$. The $\measuredangle\ \phi$ between the RLs. equals the $\measuredangle\ a_1 - a_2$ between the $\perp$s p_1, p_2 let fall on them from the origin. By Art. 25,

$$\cos a_1 = l_1 \sin\omega : \sqrt{l_1^2 + m_1^2 - 2\,l_1 m_1 \cos\omega},$$

$$\cos a_2 = l_2 \sin\omega : \sqrt{l_2^2 + m_2^2 - 2\,l_2 m_2 \cos\omega}.$$

Square, take from 1, extract square roots; there results,

$$\sin a_1 = \overline{m_1 - l_1 \cos\omega} : \sqrt{l_1^2 + m_1^2 - 2\, l_1 m_1 \cos\omega},$$

$$\sin a_2 = \overline{m_2 - l_2 \cos\omega} : \sqrt{(2)}.$$

From applying the Addition-Theorems of Sine and Cosine there results:

$$\sin\phi = \sin(a_1 - a_2) = (l_1 m_2 - l_2 m_1)\sin\omega : \sqrt{(1)}\cdot\sqrt{(2)}\,;$$

$$\cos\phi = \cos(a_1 - a_2)$$
$$= \{l_1 l_2 + m_1 m_2 - \overline{l_1 m_2 + l_2 m_1}\cos\omega\} : \sqrt{(1)}\cdot\sqrt{(2)}\,;$$

$$\tan\phi = \tan(a_1 - a_2)$$
$$= (l_1 m_2 - l_2 m_1)\sin\omega : \{l_1 l_2 + m_1 m_2 - \overline{l_1 m_2 + l_2 m_1}\cos\omega\}.$$

SPECIAL CASES. **1.** If the RLs. be $\parallel$, $\phi = 0$, $\therefore \tan\phi = 0$, $\therefore l_1 m_2 - l_2 m_1 = 0$; or, $l_1 : m_1 = l_2 : m_2$; or, $s_1 = s_2$; i.e.,

RLs. are $\parallel$ when, and only when, their Direction-Coefficients are equal.

2. If the RLs. are $\perp$, $\phi = 90°$, $\therefore \tan\phi = \infty$,

$$\therefore l_1 l_2 + m_1 m_2 - (l_1 m_2 + l_2 m_1)\cos\omega = 0. \quad \text{For } \omega = 90°,$$

$$l_1 l_2 + m_1 m_2 = 0, \text{ or } \frac{l_1}{m_1} = -\frac{m_2}{l_2}, \text{ or } s_1 = -\frac{1}{s_2}; \text{ i.e.,}$$

In rectangular Cds. two RLs. are $\perp$ when, and only when, their Direction-Coefficients are negative reciprocals of each other. This is the case when the coefficients of x and y in the one Eq. are the **exchanged** or **inverted** coefficients of x and y in the other, with the *sign of one of them changed.*

The absolute term affects neither perpendicularity nor parallelism. The rectang. Eq. of the RL. through $(x_1, y_1) \perp$ to $lx + my + n = 0$ is $l(y - y_1) = m(x - x_1)$. Why?

ILLUSTRATIONS. **1.** The sides of a $\triangle$ are: $3x - 4y + 12 = 0$, $5x + 2y + 10 = 0$, $x + 5y + 5 = 0$; find its angles ($\omega = 60°$).

$$\tan\phi_1 = \frac{(25-2)\cdot\frac{1}{2}\cdot\sqrt{3}}{5+10-(25+2)\cdot\frac{1}{2}} = \frac{23}{\sqrt{3}},$$

whence $\phi_1 = 85°\ 47'\ 28''$.

Find ϕ_2 and ϕ_3, check by $\phi_1+\phi_2+\phi_3 = 180°$, and construct the $\triangle$.

2. Find Eq. of RL. through $(1, 2)$ $\perp$ to $3x+5y=7$, and draw it, for $\omega = 90°$ and $\omega = 60°$.

27. *Distance from a point (x', y') to the RL.*

$$x\cos\alpha + y\cos\beta - p = 0.$$

The Eq. of a RL. $\parallel$ to the given RL. is

$$x\cos\alpha + y\cos\beta - p' = 0.$$

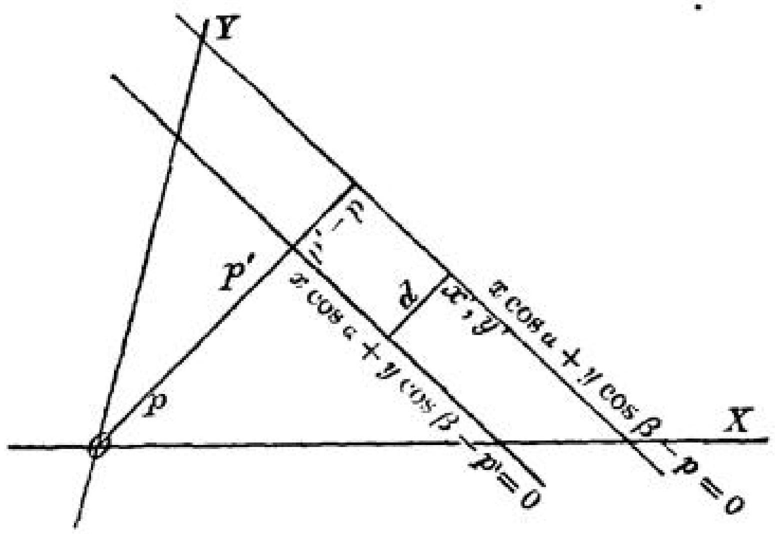
If this RL. goes through (x', y'), then

$$x'\cos\alpha + y'\cos\beta = p'.$$

Subtracting $p = p$, we get

$$x'\cos\alpha + y'\cos\beta - p = p' - p,$$

which is clearly the distance sought.



This result is + or − according as (x', y') lies on the outer or inner side of the RL. (the inner side being next to the origin). Hence the Rule: *Reduce the Eq. to the N.F., put for the current Cds. the Cds. of the point;* the result is the metric number of the distance from the point to the RL., and is + or − according as the point lies on the *outer* or *inner* side of the RL. If we put $N \equiv x \cos\alpha + y \cos\beta - p$, then **N is the distance of (x, y) from N = 0.** N changes sign, passing through 0, as (x, y) changes sides, passing through $N = 0$.

Carefully distinguish between the *expression* N and the Eq. $N = 0$. In N, x and y are Cds. of any point in the plane; in $N = 0$, x and y are Cds. of any point on the RL. $N = 0$. It is this double use, bewilder though it may at first, that gives the N.F. its importance.

ILLUSTRATIONS. **1.** Find the distance from (3, −4) to $4x + 2y = 7$.

For $\omega = 90°$, N.F. is $\dfrac{2x}{\sqrt{5}} + \dfrac{y}{\sqrt{5}} - \dfrac{7}{2\sqrt{5}} = 0$; hence, the distance is $-3 : 2\sqrt{5}$.

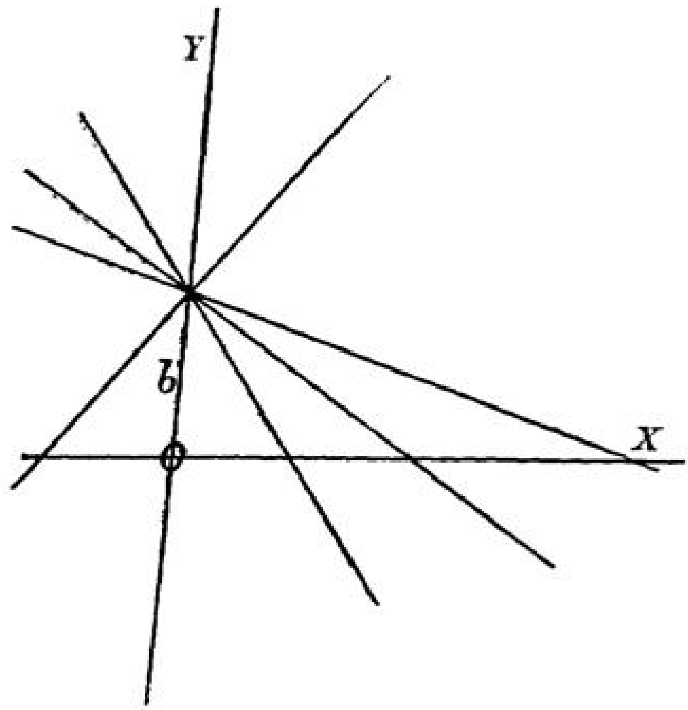
For $\omega = 60°$, the distance is $-\frac{3}{4}$. The point is on the inner side of the RL.

2. Find the distance from $(2, 3)$ to $2x + y = 4$.

3. Find the distance from the origin to $a(x - a) + b(y - b) = 0$.

The Right Line under Conditions.

28. By Art. 25 the Eq. of the RL. contains two *arbitraries* or *parameters*. If we hold one of these fast, for every value of the other we get a RL., and for the totality of values from $-\infty$ to $+\infty$ we get a family or system of RLs. Thus, in $y = sx + b$, holding b fast and assigning s the whole series of real values from $-\infty$ to $+\infty$, we get a **family** of RLs., all cutting the Y-axis b from the origin. Loosing b, assigning it the same series of values, we get a *family* of *families*, one through every point of the Y-axis. Except this Y-axis, which is common to all the families, no RL. of one family is a RL. of another: all are different RLs.; i.e., all possible different RLs. in a plane, enough to *fill* the plane, form a *family* of *families*, an *infinity* of *infinities* of RLs.; i.e., *the plane viewed as full of RLs. is doubly* *extended.* Hence, to know a RL., we must know two things about it: what *member* of what *family* it is; and to fix a RL. we must put it under *two conditions:* we must determine the *two parameters* in its Eq. We consider here some simplest cases.



I. *To find the Eq. of a family of RLs. through a point* (x_1, y_1).

The Eq. of any RL. is $y = sx + b$, and the Eq. $y_1 = sx_1 + b$ says that the RL. passes through (x_1, y_1). This last is not an Eq. of a line, since it contains no current Cds., but an Eq. of condition. By its help we can eliminate s or b, better b, and get

$$y - y_1 = s(x - x_1).$$

For any one value of s this is the Eq. of a RL. (since it contains current Cds. in 1st degree only) through (x_1, y_1), since that pair of values satisfies it. To any slope θ of such a RL. there corresponds a value of s; viz., $s = \sin\theta : \sin\overline{\omega - \theta}$, and conversely; hence, letting s range from $-\infty$ to $+\infty$, we get all RLs., the family of RLs., through (x_1, y_1).

To determine a member of this family, we may impose various conditions; as, that it go through $(x_2 y_2)$. Hence,

II. *To find the Eq. of a RL. through two points.*

Since it goes through both (x_1, y_1) and (x_2, y_2), it is common to the families

$$y - y_1 = s(x - x_1) \quad \text{and} \quad y - y_2 = s(x - x_2),$$

and for *it* s has the same value in the two Eqs.; hence, eliminating s, we get the Eq. sought:

$$y - y_1 : y - y_2 = x - x_1 : x - x_2.$$

Let the student interpret this proportion geometrically.

We may reason otherwise; thus:

Be $lx + my + n = 0$ the Eq. of the RL.; since it goes through (x_1, y_1),

$$lx_1 + my_1 + n = 0;$$

and, for like reason,

$$lx_2 + my_2 + n = 0.$$

By Introduction, Art. 14, these Eqs. consist when, and only when,

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

This Eq. of the RL. is equivalent to the other, and very convenient.

Corollary. $\dfrac{y_3 - y_1}{y_3 - y_2} = \dfrac{x_3 - x_1}{x_3 - x_2}$, or $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

is the Eq. of condition that (x_1, y_1), (x_2, y_2), (x_3, y_3) lie on a RL.

III. *To find the Eq. of a RL. through a given point* (x_1, y_1) *and sloped* ϕ *to a RL. whose Direction-coefficient is* s_1 $(\omega = 90°)$.

The RL. is of the family $y - y_1 = s(x - x_1)$; also, by Art. 26,

$$\tan\phi = \frac{lm_1 - l_1 m}{ll_1 + mm_1} = \frac{s_1 - s}{1 + ss_1}, \quad \text{or} \quad s = \frac{s_1 - \tan\phi}{1 + s_1 \tan\phi};$$

$$\therefore y - y_1 = \frac{s_1 - \tan\phi}{1 + s_1 \tan\phi}(x - x_1).$$

Corollary. If $\phi = 90°$, i.e., if the RL. is to be ⊥ to the given RL.,

$$y - y_1 = -\frac{1}{s_1}(x - x_1) \quad \text{or} \quad l_1(y - y_1) = m_1(x - x_1).$$

Let the student solve this problem for ω *not* $= 90°$.

EXERCISES.

1. The vertices of a △ are $(5, -7)$, $(1, 11)$, $(-4, 13)$; find the Eqs. of: its sides, its medials, ⊥s through the mid-points of its sides, ⊥s from its vertices on the counter sides; find the lengths of the last ⊥s.

2. Find the Eq. of the RL. through (x_3, y_3) cutting $\overline{P_1P_2}$ in the ratio $\mu_1 : \mu_2$.

3. Find the RL. through $(5,4)$ forming with + axes a △ of area 80. $(\omega = 90°)$.

4. The sides of a 4-side, taken in order, are $y = 5x$, $6y + 5x = 35$, $3x - y = 21$, $4y + 9x = 0$; find the Cds. of its vertices, Eqs. of its diagonals, and of the junction-lines of their mid-points.

5. Find Eqs. of RLs. through the junction-point of $3x - 4y = 7$ and $2x + 5y + 8 = 0$, and sloped 60° to $y = 4x + 3$.

6. Find Eqs. of RLs. sloped 30° to X-axis and cutting Y-axis 7.5 from origin.

7. Are $(6,2)$, $(7,-3)$, $(-5,-5)$, on a RL.? Are $(3,-1)$, $(1,2)$, $(7,-7)$?

8. Two counter sides of a 4-side being axes, the other sides are $\frac{x}{2a'} + \frac{y}{2b} = 1$, $\frac{x}{2a'} + \frac{y}{2b} = 1$; find the mid-points of the diagonals.

29. The *parameter* of a *family* of RLs. is the *arbitrary* in its Eq.

Theorem. *When the parameter appears in 1st degree only, all the RLs. of the family go through a point.*

Solve the Eq. as to the parameter λ, we get

$$\lambda = (l_1x + m_1y + n_1) : (l_2x + m_2y + n_2),$$

or $$l_1x + m_1y + n_1 - \lambda(l_2x + m_2y + n_2) = 0,$$

or $$L_1 - \lambda L_2 = 0.$$

Whatever λ be, this RL. goes through the junction-point of the RLs. $L_1 = 0$ and $L_2 = 0$, since the pair (x, y) which satisfies both these Eqs. also satisfies $L_1 - \lambda L_2 = 0$.

Conversely, *all RLs. through the junction-point of* $L_1 = 0$ and $L_2 = 0$ *are of the family* $L_1 - \lambda L_2 = 0$. For the Direction-coefficient is $s = (\lambda l_2 - l_1) : (m_1 - \lambda m_2)$, and this ranges with λ through all real values.

Hence, to find the Eq. of any special RL. through the junction of $L_1 = 0$ and $L_2 = 0$, which let us call the point $(L_1 . L_2)$, it suffices to find the corresponding special value of λ, which we may call the parameter of that RL. Thus, if the RL. is to pass through the origin, then $n_1 - \lambda n_2 = 0$, $\lambda = n_1 : n_2$; if it is to be $\parallel$ to the X- resp. Y-axis, then $l_1 - \lambda l_2 = 0$ resp. $m_1 - \lambda m_2 = 0$, $\lambda = l_1 : l_2$ resp. $\lambda = m_1 : m_2$.

The student may substitute these values of λ, and find the Eqs. of the RLs.

30. The above is a special case of the important theorem: the curve $C_1 - \lambda C_2 = 0$ goes through all junction-points of the curves $C_1 = 0$, $C_2 = 0$ (λ being any constant). For any pair (x, y) that satisfies both $C_1 = 0$ and $C_2 = 0$, satisfies also $C_1 - \lambda C_2 = 0$.

This again is a special case of the still higher theorem: If, when two Eqs. are satisfied, a third is also satisfied, the third curve passes through all junction-points of the other two.

The proposition is evident as soon as its terms are understood.

Hence, it is plain that the RL. $\mu_1 L_1 + \mu_2 L_2 = 0$ goes through the junction of $L_1 = 0$ and $L_2 = 0$, since its Eq. is satisfied when the others are.

If the expression $\mu_1 L_1 + \mu_2 L_2 + \mu_3 L_3 \equiv 0$, i.e., vanish identically, then the three RLs. $L_1 = 0$, $L_2 = 0$, $L_3 = 0$, pass through a point, since the pair that satisfies two of the Eqs. must satisfy the third. Hence, if the sum of some multiples of the Eqs. of three RLs., vanish identically, the three RLs. go through a point.

If $\mu_1 L_1 + \mu_2 L_2 + \mu_3 L_3 = 0$, but not identically, then this Eq. imposes some condition on the symbols x and y; hence it is the Eq. of some line, and in fact of some RL., since x and y enter the Eq. in first degree only. We may go further, and say that this Eq. may be made the Eq. of any RL. by choosing the μ's properly. For $lx + my + n = 0$ is any RL., and we may choose the three μ's so as to satisfy the three Eqs.:

$$\mu_1 l_1 + \mu_2 l_2 + \mu_3 l_3 = l,$$

$$\mu_1 m_1 + \mu_2 m_2 + \mu_3 m_3 = m,$$

$$\mu_1 n_1 + \mu_2 n_2 + \mu_3 n_3 = n.$$

The RLs. $L_1 = 0$, $L_2 = 0$, $L_3 = 0$, determine a △, which may be called the △ *of reference*, or **referee** △; and the expressions L_1, L_2, L_3, may be called the *trilinear*, or **triangular** Cds. of points on the RL. $\mu_1 L_1 + \mu_2 L_2 + \mu_3 L_3 = 0$.

If F_1, F_2, F_3, be the normalizing factors of L_1, L_2, L_3, then $F_1 L_1$, $F_2 L_2$, $F_3 L_3$, are the distances of any point (x, y) from the RLs. $L_1 = 0$, $L_2 = 0$, $L_3 = 0$. Hence, it is seen that the *triangular Cds. of a point* (x, y) *are certain fixed multiples of its distances from the sides of the referee* △. Triangular Cds. will be simplest, then, when they are the simplest multiples of the distances from the sides of the △; i.e., when they are the distances themselves; and these they are when, and only when, the Eqs. of the sides of the △ are in the N.F.; in the N.F. we write them, $N_1 = 0$, $N_2 = 0$, $N_3 = 0$, where

$$N_k \equiv x \cos \alpha_k + y \cos \beta_k - p_k.$$

Hence, calling now the multipliers of the N's ν's, we have

$$\nu_1 N_1 + \nu_2 N_2 + \nu_3 N_3 = 0$$

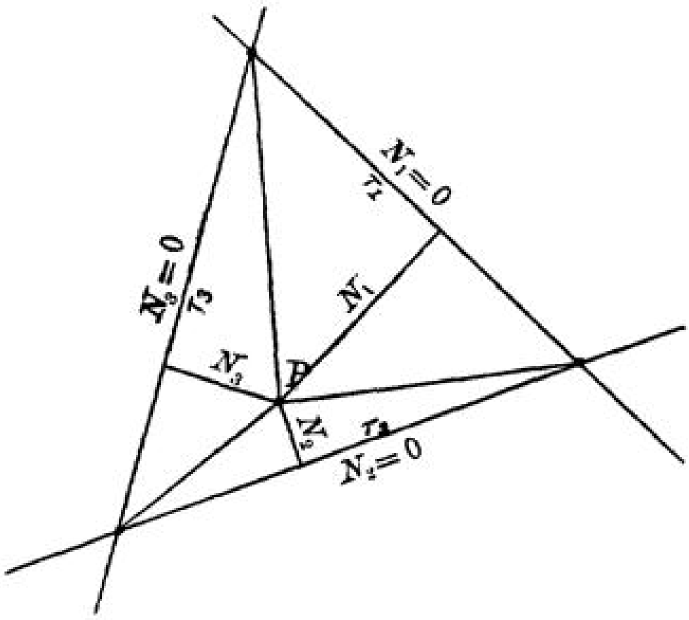
as the *normal Eq. of a RL. in triangular Cds.:* N_1, N_2, N_3.

31. Here the point has *three* Cds., the N's, while we know that *two* are enough to fix it. But, since the Eq. is homogeneous in N's, we can at once reduce the number of Cds. to two by dividing through by one, as N_3, and treating the quotients $N_1 : N_3$,

$N_2 : N_3$ as the independent Cds. In general, the *triangular Cds. of a point are bound together by a certain constant relation.* For if we suppose, as we may, that the origin O is within the referee $\triangle$, and put τ_1, τ_2, τ_3 for the tracts between the vertices, A for its area, then for every point (x, y) or (N_1, N_2, N_3) we shall have

$$\tau_1 N_1 + \tau_2 N_2 + \tau_3 N_3 = -2A.$$

This is clear at once when P is within the $\triangle$, and is equally clear, on proper regard of signs, when P is without. By multiplying this Eq. appropriately, we can express any constant homogeneously through the triangular Cds. of any point. If the origin O be without the $\triangle$, it suffices to change the sign of one of the N's.



A general test of whether three RLs. $L_1 = 0$, $L_2 = 0$, $L_3 = 0$, go through a point, is found in the Determinant

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix},$$

which, by Introduction, Art. 14, must vanish when the three Eqs.

$$l_1 x + m_1 y + n_1 = 0,$$

$$l_2 x + m_2 y + n_2 = 0,$$

and

$$l_3 x + m_3 y + n_3 = 0,$$

consist, are all satisfied by the same pair (x, y).

So too must $\begin{vmatrix} \nu_1' & \nu_2' & \nu_3' \\ \nu_1'' & \nu_2'' & \nu_3'' \\ \nu_1''' & \nu_2''' & \nu_3''' \end{vmatrix} = 0,$

when through a point go the RLs.

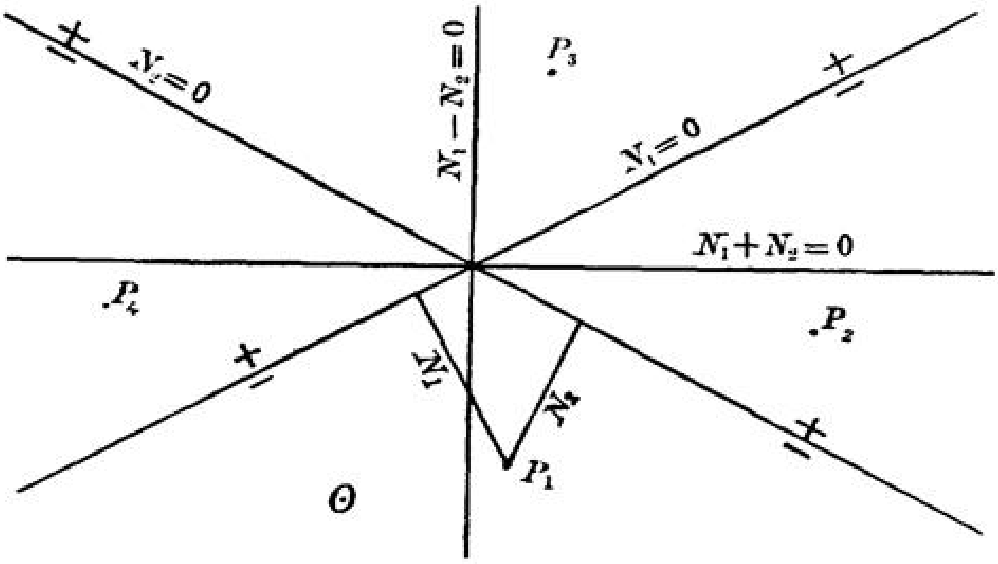
$$\nu_1' N_1 + \nu_2' N_2 + \nu_3' N_3 = 0,$$

$$\nu_1'' N_1 + \nu_2'' N_2 + \nu_3'' N_3 = 0,$$

$$\nu_1''' N_1 + \nu_2''' N_2 + \nu_3''' N_3 = 0.$$

Geometric Interpretation of λ.

32. A family of RLs. through a point we may call a **pencil**, the point itself, the **centre** of the pencil; the two RLs. through which the others are expressed, the **base-lines** of the pencil. If the Eqs. of the base-lines be in the N.F., then $N_1 - \lambda N_2 = 0$, for any special value of λ, is the Eq. of a RL. of the pencil.



Here $\lambda = N_1 : N_2 =$ *ratio of the distances of any point of the RL.* $N_1 - \lambda N_2 = 0$ *from the base-lines* $N_1 = 0$ and $N_2 = 0$; or, $\lambda =$ *ratio of the sines of the slopes of the RL.* $N_1 - \lambda N_2 = 0$ *to the base-lines* $N_1 = 0$ *and* $N_2 = 0$.

If we call the angle containing the origin, and its vertical angle, and the lines in them, all *inner*, the others all *outer*, then we see that for inner lines the distances are like-signed, as from P_1, P_3, and $\therefore \lambda$ is $+$; for outer lines the distances are unlike-signed, as from P_2, P_4, and $\therefore \lambda$ is -

If we call the *inner* side of a RL. also the $-$ side, the *outer* also the $+$ side, then angles and sines of angles reckoned from the $+$ resp. $-$ side of a RL. are themselves $+$ resp. $-$; we agree to reckon angles from the fixed to the variable RL.

33. For the inner resp. outer halvers of the angles at (N_1, N_2) the distances are equal and like- resp. unlike-signed; hence, $\lambda = +1$ resp. -1; hence, the halvers are $N_1 - N_2 = 0$ resp. $N_1 + N_2 = 0$.

Hence, to find the Eq. of the *inner* resp. *outer* halver of the ∡s between two RLs., form the *difference* resp. *sum* of their Eqs. in the N.F.

N.B. Of the two equivalent Eqs., $\pm N = 0$, we have taken *that* as the N.F. in which the absolute term is negative. But this test fails, and with it the test of which is the $+$ and which the $-$ side of the RL., when the absolute term is 0, i.e., when the RL. goes through the origin. In this case, we may agree to take always the term in y, or always that in x, as positive. If we make the first agreement, as is common, the $+$ side will lie next to the $+\,Y$-axis; for, holding x and letting y increase, we must get a $+$ result in the left member of the Eq.

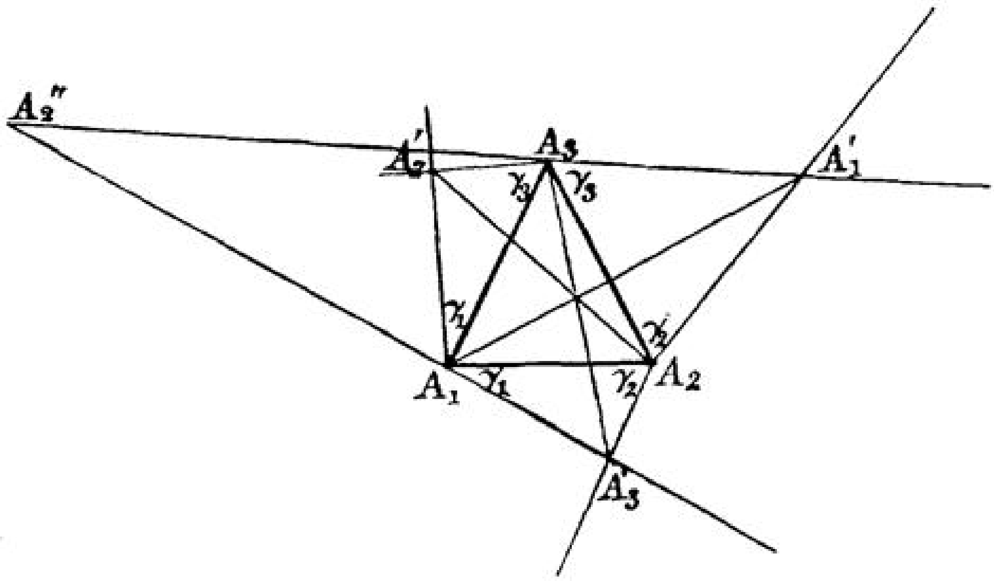
34. This **Abridged Notation** (a single letter N standing for the left member of the Eq. of a RL. in the N.F.) with its immediate outgrowth, the system of **triangular** Cds., yields a method of great strength and beauty; we have space for but a few simple

Illustrations. **1.** The inner halvers of the ∡s of a △ meet in a point. For, be the origin within the △, and $N_1 = 0$, $N_2 = 0$, $N_3 = 0$, its sides, then the sum of the Eqs. of the inner halvers, $N_1 - N_2 = 0$, $N_2 - N_3 = 0$, $N_3 - N_1 = 0$, vanishes identically.

2. Two outer halvers and the third inner halver meet in a point. They are $N_2 + N_3 = 0$, $N_3 + N_1 = 0$, $N_1 - N_2 = 0$; multiply the second by -1, and sum.

3. The ⊥s from the vertices of a △ on the counter sides meet in a point. The ∡s being A_1, A_2, A_3, the ⊥ from A_1 on $N_1 = 0$ is $N_2 \cos A_2 - N_3 \cos A_3 = 0$; permute the indices, and sum.

4. If through each vertex of a △ be drawn a pair of rays like-sloped to the sides meeting there, the junction-lines of the junction-point of each couple next to a side and the counter vertex meet in a point.



Be $N_1 = 0$, $N_2 = 0$, $N_3 = 0$ the sides, A_1, A_2, A_3 the counter vertices, γ_1, γ_2, γ_3 the slopes of the pairs at the vertices. The Eqs. of A_2A_1' resp. A_3A_1' are

$$N_1 \sin(\gamma_2 + A_2) + N_3 \sin \gamma_2 = 0,$$

resp. $$N_1 \sin(\gamma_3 + A_3) + N_2 \sin \gamma_3 = 0.$$

The Eq. of A_1A_1' is, therefore,

$$N_1 \sin(\gamma_2 + A_2) + N_3 \sin \gamma_2 - \lambda\{N_1 \sin \overline{\gamma_3 + A_3} + N_2 \sin \gamma_3\} = 0.$$

At A_1 both N_2 and N_3 are 0; hence,

$$\lambda = \sin \overline{\gamma_2 + A_2} : \sin \overline{\gamma_3 + A_3}.$$

$$\therefore\ N_3 \cdot \sin \gamma_2 \cdot \sin \overline{\gamma_3 + A_3} - N_2 \cdot \sin \gamma_3 \cdot \sin \overline{\gamma_2 + A_2} = 0$$

is the Eq. of A_1A_1';

$$N_1 \cdot \sin \gamma_3 \cdot \sin \overline{\gamma_1 + A_1} - N_3 \cdot \sin \gamma_1 \cdot \sin \overline{\gamma_3 + A_3} = 0$$

is the Eq. of A_2A_2';

$$N_2 \cdot \sin \gamma_1 \cdot \sin \overline{\gamma_2 + A_2} - N_1 \cdot \sin \gamma_2 \cdot \sin \overline{\gamma_1 + A_1} = 0$$

is the Eq. of $A_3 A_3'$.

Multiply by $\sin \gamma_1$, $\sin \gamma_2$ resp. $\sin \gamma_3$, and sum.

If $A_3 A_1'$ meets $A_1 A_3'$ in A_2'', show that $A_2 A_2''$, $A_3 A_3''$, $A_1 A_1''$ meet in a point.

5. If through each vertex of a △ be drawn a pair of rays like-sloped to the halver of the ∡ at that vertex, and if any three of these rays meet in a point, so will the other three.

If $N_1 \sin \overline{A_2 + \gamma_2} - N_3 \sin \gamma_2 = 0$, $N_2 \sin \overline{A_3 + \gamma_3} - N_1 \sin \gamma_3 = 0$, $N_3 \sin \overline{A_1 + \gamma_1} - N_2 \sin \gamma_1 = 0$, be any three, the others are got by simply exchanging the N's in each Eq. The value of any N in one of these Eqs. is not in general the same as its value in another; but if the first three RLs. meet in a point, the first three Eqs. must consist for one set of values of the N's; hence we may transpose, multiply, cancel the product $N_1 N_2 N_3$, and get as a condition that the three RLs. meet in a point, the Eq.,

$$\sin \overline{A_2 + \gamma_2} \cdot \sin \overline{A_3 + \gamma_3} \cdot \sin \overline{A_1 + \gamma_1} = \sin \gamma_2 \cdot \sin \gamma_3 \cdot \sin \gamma_1.$$

Now, clearly, exchanging the N's in each Eq. affects not this result.

6. If the three ⊥s from the three vertices of one △ on the three sides of a second △ meet in a point, so do the three ⊥s from the three vertices of the second on the three sides of the first.

Be the sides of the one △ $N_1 = 0$, $N_2 = 0$, $N_3 = 0$, the sides of the other $N_1' = 0$, $N_2' = 0$, $N_3' = 0$; then

$$N_2 \cos \widehat{N_1' N_3} = N_3 \cdot \cos \widehat{N_1' N_2}, \quad N_2' \cos \widehat{N_1 N_3'} = N_3' \cdot \cos \widehat{N_1 N_2'}$$

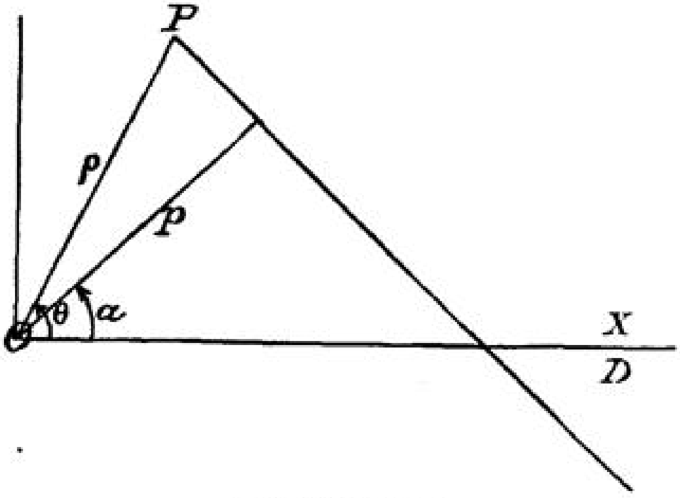
are a pair of corresponding ⊥s. Let the student complete the proof.

Polar Equation of the Right Line.

35. If the ⊥ from the origin on the RL. be sloped α to the polar axis OD, we have at once,

$$\rho\cos(\theta-\alpha)=p$$

Let the student get this Eq. by transforming to polar Cds. from the N.F.



EXERCISES.

1. Reduce $\rho=2a\sec\left(\theta+\frac{\pi}{6}\right)$ to rectang. Cds.

2. Find where $\rho=a\sec\left(\theta-\frac{\pi}{6}\right)$ and $\rho\cos\left(\theta-\frac{\pi}{2}\right)=2a$ meet, and under what angle.

3. Find where the ⊥ from the pole O meets the RL. through (ρ_1,θ_1), (ρ_2,θ_2).

Miscellany.

36. The Eq. of the RL. through (x_2,y_2), (x_3,y_3), in the N.F. is

$$\begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\sin\omega:\sqrt{\{\overline{y_2-y_3}^2+\overline{x_2-x_3}^2+2\,\overline{y_2-y_3}\cdot\overline{x_2-x_3}\cdot\cos\omega\}}=0.$$

The left side is the distance from (x,y) to the RL., the $\sqrt{}$ is the length of the tract $\overset{\longmapsto}{(x_2,y_2)(x_3,y_3)}$; hence, their product, (x_1,y_1), being written for (x,y), or

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\sin\omega,$$

is the *double area of the* △ whose vertices are (x_1,y_1), (x_2,y_2), (x_3,y_3).

37. *To find the area of a polygon* whose vertices (taken in order) are

$$(x_1, y_1),\ (x_2, y_2)\ \cdots\ (x_n, y_n).$$

Note that when one vertex of a $\triangle$ is the origin $(0, 0)$, its double area is $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$, (x_1, y_1), (x_2, y_2) being the other vertices, the axes assumed rectangular. Now from the origin, assumed within the polygon, draw rays to the vertices cutting it up into n $\triangle$; the double area of the polygon then is

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \cdots + \begin{vmatrix} x_{n-1} & x_n \\ y_{n-1} & y_n \end{vmatrix} + \begin{vmatrix} x_n & x_1 \\ y_n & y_1 \end{vmatrix}.$$

Now suppose the origin moved to (x_0, y_0); the double area of the first $\triangle$ changes to

$$\begin{vmatrix} x_1'+x_0, & x_2'+x_0 \\ y_1'+y_0, & y_2'+y_0 \end{vmatrix} \equiv \begin{vmatrix} x_1' & x_2' \\ y_1' & y_2' \end{vmatrix} + \begin{vmatrix} x_1' & x_0 \\ y_1' & y_0 \end{vmatrix} + \begin{vmatrix} x_0 & x_2' \\ y_0 & y_2' \end{vmatrix} + \begin{vmatrix} x_0 & x_0 \\ y_0 & y_0 \end{vmatrix}.$$

The first of these Determinants is the original primed, the last vanishes, the sum of second and third is

$$x_0(y_2' - y_1') + y_0(x_1' - x_2').$$

If we operate likewise upon the other Determinants, we shall clearly get the original Determinants primed, while the sum of the multipliers of x_0 and y_0 will each vanish identically. Hence wherever the origin be, the double polygonal area is

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \cdots + \begin{vmatrix} x_n & x_1 \\ y_n & y_1 \end{vmatrix},$$

the primes being dropped.

We thus learn that the *sum of the areas of* $\triangle$ *with one common vertex, the other vertices being vertices of a polygon, is the area of that polygon:* a theorem important in mechanics. The practical rule is to write the x's and y's in order, each in a horizontal row, repeating the first of each as the last; thus,

$$\begin{matrix} x_1 & x_2 & x_3 & \cdots & x_n & x_1 \\ y_1 & y_2 & y_3 & \cdots & y_n & y_1 \end{matrix};$$

and take the cross-products of the consecutive pairs with opposite signs. The algebraic fact that the sign of the area is + or − according as we take one or the other set of products +, corresponds to the geometric fact that we may compass the polygon *clockwise* or *counter-clockwise*.

For axes not rectangular, multiply by $\sin \omega$.

38. *To find the area of the* $\triangle$ *whose sides are* $L_1 = 0$, $L_2 = 0$, $L_3 = 0$, form the Determinant of the coefficients:

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix},$$

and denote the co-factor of any element, as l_1, by that symbol primed, l_1'; be (x_1, y_1) the junction-point of $L_2 = 0$, $L_3 = 0$. Then we have $x_1 = l_1' : n_1'$, $y_1 = m_1' : n_1'$, and so for (x_2, y_2), (x_3, y_3). Form the Determinant of Art. 36; replace the column of 1's by $n_1' : n_1'$, $n_2' : n_2'$, $n_3' : n_3'$; set out the divisors of the rows, n_1', n_2', n_3'; there results

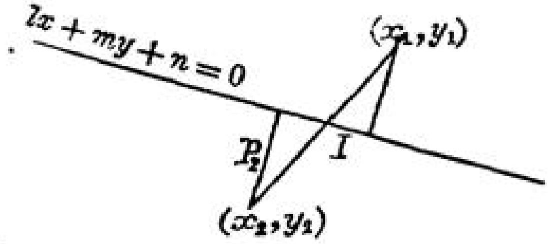
$$2\triangle = | l_1' m_2' n_3' | \sin \omega : n_1' \cdot n_2' \cdot n_3'$$
$$= | l_1 m_2 n_3 |^2 \sin \omega : | l_1 m_2 | \cdot l_2 m_3 | \cdot l_3 m_1 |.$$

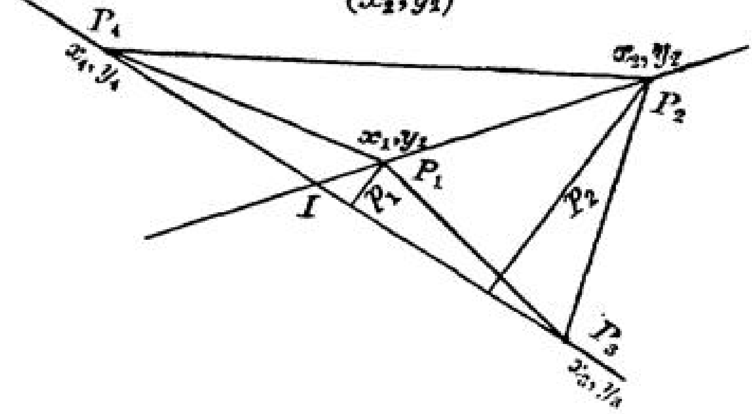
If two of the RLs. be ||, a factor of the denominator vanishes, the area is ∞; if the three RLs. meet in a point, the numerator vanishes, the area is 0.

39. *To find the ratio* $\mu_1 : \mu_2$ *in which the tract* $\overline{(x_1, y_1)(x_2, y_2)}$ *is cut by the RL.* $lx + my + n = 0$. We have at once, on cancelling the normalizing factor F,

$$\frac{\mu_1}{\mu_2} = -\frac{p_1}{p_2} = -\frac{lx_1 + my_1 + n}{lx_2 + my_2 + n}.$$

When the section is inner, μ_1 and μ_2 are like-signed, p_1 and p_2 unlike-signed; when the section is outer, the μ's are unlike-signed, the p's like-signed; hence the − signs in the above Eq.





If the RL. go through (x_3, y_3) and (x_4, y_4), its Eq. is

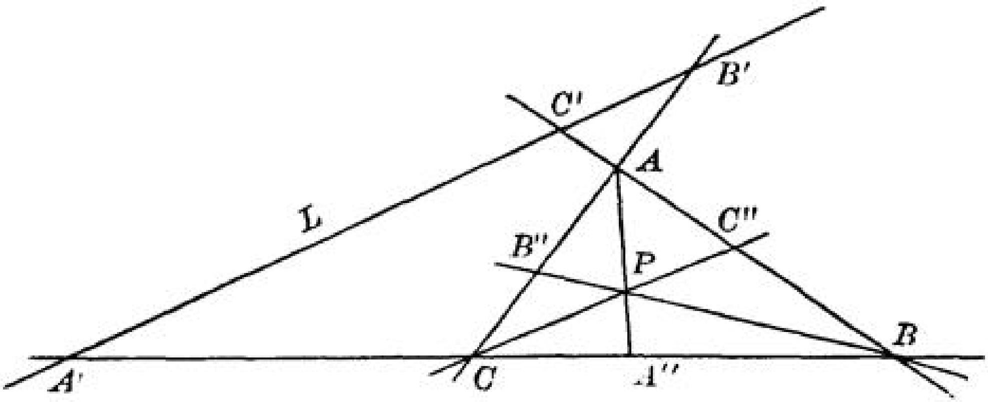
$$\begin{vmatrix} x & y & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} = 0; \text{ hence, } \mu_1 : \mu_2 = -\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} : \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$$

$$= -\frac{P_1P_3P_4}{P_2P_3P_4}.$$

40. By the help of these expressions for $\mu_1 : \mu_2$ we can now prove two fundamental Theorems on Transversals:

I. *Three points on a RL. and the sides of a △ cut its sides into segments whose compound ratio is* -1.

II. *Three RLs. through a point and the vertices of a △ cut its sides into segments whose compound ratio is* $+1$.



Be ABC the △, $L \equiv lx + my + n = 0$ the transversal, cutting the sides at A', B', C'; be $_kL \equiv lx_k + my_k + n$ the result of putting x_k, y_k in L for x, y. Then, by Art. 39,

$$BA' : A'C = -{}_2L : {}_3L,$$
$$CB' : B'A = -{}_3L : {}_1L,$$
$$AC' : C'B = -{}_1L : {}_2L;$$

the product of these ratios is -1, which proves I.

Be (x, y) the Cds. of P, through which are drawn AA'', BB'', CC''. Then

$$\frac{BA''}{A''C} = -\begin{vmatrix} x_2 & y_2 & 1 \\ x_1 & y_1 & 1 \\ x & y & 1 \end{vmatrix} : \begin{vmatrix} x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \\ x & y & 1 \end{vmatrix},$$

$$\frac{CB''}{B''A} = -\begin{vmatrix} x_3 & y_3 & 1 \\ x_2 & y_2 & 1 \\ x & y & 1 \end{vmatrix} : \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x & y & 1 \end{vmatrix},$$

$$\frac{AC''}{C''B} = -\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x & y & 1 \end{vmatrix} : \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x & y & 1 \end{vmatrix}.$$

In the product of these ratios the determinants of the denominator are those of the numerator with two rows exchanged; i.e., with signs changed; hence, the product is $+1$, which proves II.

Note the order of letters and indices.

EXERCISES.

1. Find the areas of the △ whose vertices are: $(2, 1)$, $(3, -2)$, $(4, -1)$; $(2, 3)$, $(4, -5)$, $(-3, -6)$. *Ans.* 10; 29.

2. Find the area of the tetragon: $(1, 1)$, $(2, 3)$, $(3, 3)$, $(4, 1)$.

Solution. $\begin{matrix} 1 & 2 & 3 & 4 & 1 \\ 1 & 3 & 3 & 1 & 1 \end{matrix}$, $3 + 6 + 3 + 4 - 2 - 9 - 12 - 1 = -8$; ∴ *Ans.* 4.

3. Find the area of the △: $2x - 3y = 5$, $4x + 5y = -7$, $y - 2x = 9$.

Cross Ratios.

41. A set of points on a RL. is called a *range* or *row;* a set of RLs. through a point, a *pencil;* each (half-) RL., a *ray*.

The RL. is the carrier of the row or range; the point, the centre or carrier of the pencil.

The *distance-ratio* of P_2 to P_1 and P_3 is $P_1P_2 : P_2P_3$ (Art. 17); and so the *distance-ratio* of P_4 to P_1 and P_3 is $P_1P_4 : P_4P_3$.

The **ratio** *of the distance-ratios of two points of a range to two other points of that range is called the* **cross ratio** *of the range*, and is written

$$\{P_1P_2P_3P_4\} = \frac{P_1P_2}{P_2P_3} : \frac{P_1P_4}{P_4P_3}.$$

Observing signs, we see at once,

$$\{P_1P_2P_3P_4\} = \frac{P_1P_2 \cdot P_3P_4}{P_2P_3 \cdot P_4P_1},$$

or, simply, $\{1\ 2\ 3\ 4\} = \dfrac{\overline{1\ 2} \cdot \overline{3\ 4}}{\overline{2\ 3} \cdot \overline{4\ 1}}$,

which is the neatest way to write it.

Clearly, the order of the points is essential. The *alternates in position*, as 1st and 3d, 2d and 4th, are called **conjugates;** the *consecutives* are **non-conjugates**; as 1st and 2d, 2d and 3d, 3d and 4th, 4th and 1st. Any one has one conjugate, two non-conjugates.

The four symbols may be permuted in 4!, = 24, ways. Any permutation or order may be got from any other, as 1 2 3 4, by one or more exchanges, which will be either of a pair of conjugates or of a pair of non-conjugates.

To exchange a pair of conjugates **inverts** *the ratio.*

Be $$\{1\ 2\ 3\ 4\} = \frac{\overline{1\ 2} \cdot \overline{3\ 4}}{\overline{2\ 3} \cdot \overline{4\ 1}} = r;$$

then $$\{1\ 4\ 3\ 2\} = \frac{\overline{1\ 4} \cdot \overline{3\ 2}}{\overline{4\ 3} \cdot \overline{2\ 1}} = \frac{\overline{2\ 3} \cdot \overline{4\ 1}}{\overline{1\ 2} \cdot \overline{3\ 4}} = \frac{1}{r}.$$

To exchange a pair of non-conjugates takes the **complement** *of the ratio to* 1. For, exchange (say) 2 and 3; then

$$\{1\,3\,2\,4\}=\frac{\overline{1\,3}\cdot\overline{2\,4}}{\overline{3\,2}\cdot\overline{4\,1}}=\frac{(\overline{1\,2}+\overline{2\,3})\cdot(\overline{2\,3}+\overline{3\,4})}{\overline{3\,2}\cdot\overline{4\,1}}$$

$$=\frac{\overline{1\,2}\cdot\overline{3\,4}}{\overline{3\,2}\cdot\overline{4\,1}}+\frac{\overline{2\,3}(\overline{1\,2}+\overline{2\,3}+\overline{3\,4})}{\overline{3\,2}\cdot\overline{4\,1}}$$

$$=-\frac{\overline{1\,2}\cdot\overline{3\,4}}{\overline{2\,3}\cdot\overline{4\,1}}+\frac{\overline{2\,3}\cdot\overline{1\,4}}{\overline{3\,2}\cdot\overline{4\,1}}=1-r.$$

Hence, to exchange two pairs of conjugates, or two pairs of non-conjugates, keeps the cross ratio unchanged. Hence there are four permutations for which the cross ratio (or CR.) is the same; hence, too, there are six permutations, or six sets of four permutations, for which the CRs. are different. By inverting and complementing to 1, we get these values:

$$r,\ \frac{1}{r},\ 1-r,\ \frac{1}{1-r},\ 1-\frac{1}{r},\ 1-\frac{1}{1-r}.$$

The circle of these six values is complete; any amount of inverting and complementing will reproduce one of them. The last two may be written

$$\frac{r-1}{r}\text{ and }\frac{r}{r-1}.$$

The ratio of the sines of the ∡s into which a third ray cuts the ∡ between two other rays is called the **sine ratio** *of the third ray to the other two; the* **ratio** *of the sine ratios of two rays to two others is called the* **cross ratio** *of the rays*, and is written

$$S\{1\,2\,3\,4\}=\frac{\sin\widehat{1\,2}}{\sin\widehat{2\,3}}:\ \frac{\sin\widehat{1\,4}}{\sin\widehat{4\,3}}=\frac{\sin\widehat{1\,2}\cdot\sin\widehat{3\,4}}{\sin\widehat{2\,3}\cdot\sin\widehat{4\,1}},$$

when S is the centre of the pencil, and the rays are 1, 2, 3, 4.

When the *rays* of a *pencil* go through the *points* of a *range*, the two are *conjoined*.

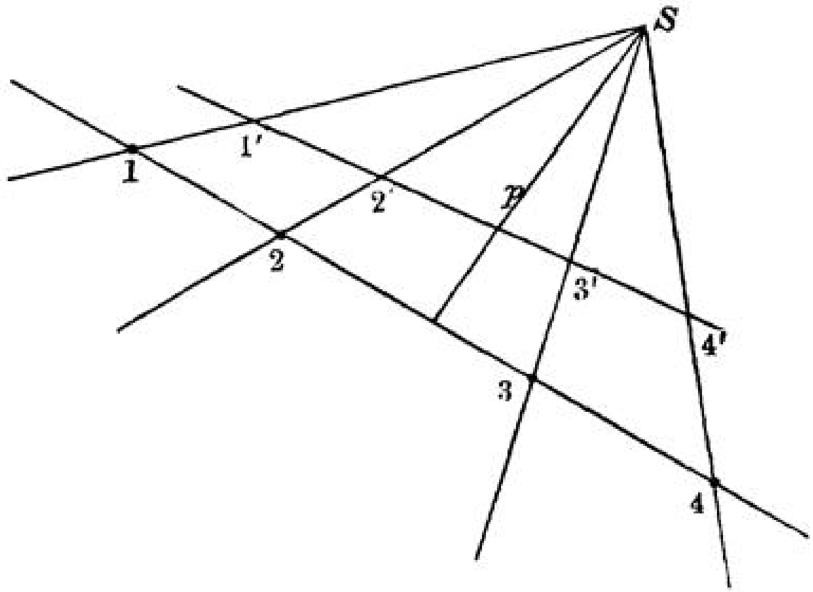
From the figure, we see at once

$$p \cdot \overline{1\ 2} = S1 \cdot S2 \cdot \sin \widehat{1\ 2}, \quad p \cdot \overline{3\ 4} = S3 \cdot S4 \cdot \sin \widehat{3\ 4},$$

$$p \cdot \overline{1\ 4} = S1 \cdot S4 \cdot \sin \widehat{1\ 4}, \quad p \cdot \overline{2\ 3} = S2 \cdot S3 \cdot \sin \widehat{2\ 3},$$

whence, $\{1\ 2\ 3\ 4\} = S\{1\ 2\ 3\ 4\}$; or

The CRs. of a conjoined range and pencil are equal.



Hence, holding the carrier of either range or pencil fast, we may move the other at will.

If $\{1\ 2\ 3\ 4\} = -1 = S\{1\ 2\ 3\ 4\}$, the tract between two conjugate points, resp. ∡ between two conjugate rays, is cut innerly and outerly in the same ratio by the other conjugates; the *range* resp. *pencil* is then called **harmonic**, and either *pair of conjugates* (points or rays) is called **harmonic** to the *other pair*, while the *fourth element* (point or ray) is called a *fourth harmonic* to the other three in order.

If in an harmonic range *one point* **halve innerly** *the tract between two conjugates, the other* (its conjugate) *must* **halve** *the same tract* **outerly,** i.e., *must be at* ∞.

If in an harmonic pencil *one ray* **halve** *innerly the* ∡ *between two conjugates, the other* (its conjugate) *must* **halve** *the same* ∡ **outerly**; i.e., *must be* ⊥ *to the first.*

Clearly, both these propositions may be converted by simply exchanging the words *innerly* and *outerly*.

Also, if of two conjugates in an harmonic range *one be at* ∞, *the other* **halves innerly** *the tract between the other pair;* and, if *one pair of conjugate rays in an harmonic be at R.* ∠s, *they* **halve innerly** *and* **outerly** *the* ∡ *of the other pair.*

EXERCISE.

Prove by similar △s that $\{1\ 2\ 3\ 4\} = \{1'\ 2'\ 3'\ 4'\}$.

42. *Cross Ratio of four rays given by their equations.*

Be the base-rays, $N_1 = 0$, $N_3 = 0$, a pair of conjugates, and be $N_2 = N_1 - \lambda_2 N_3 = 0$, $N_4 = N_1 - \lambda_4 N_3 = 0$, the other pair. Denoting the rays by their proper indices, we see at once, from Art. 32,

$$\lambda_2 = -\frac{\sin \widehat{1\ 2}}{\sin \widehat{2\ 3}},\ \ \lambda_4 = -\frac{\sin \widehat{1\ 4}}{\sin \widehat{4\ 3}};\ \text{ hence } \{1\ 2\ 3\ 4\} = \lambda_2 : \lambda_4.$$

If $L_1 = 0$, $L_3 = 0$ be the *base-rays*, f_1, f_3 their normalizing factors; then $L_1 = 0$, $L_1 - \lambda_2 L_3 = 0$, $L_3 = 0$, $L_1 - \lambda_4 L_3 = 0$ may be written in normal form; thus,

$$f_1 L_1 = 0, \quad f_1 L_1 - \lambda_2 \frac{f_1}{f_3} \cdot f_3 L_3 = 0,$$

$$f_3 L_3 = 0, \quad f_1 L_1 - \lambda_4 \frac{f_1}{f_3} \cdot f_3 L_3 = 0;$$

whence, by the above, we have again, $\{1\ 2\ 3\ 4\} = \lambda_2 : \lambda_4$.

If $L' = 0$, $L'' = 0$ be *base-rays;* then to find the CR. of any rays, $L' - \lambda_1 L'' = 0$, $L' - \lambda_2 L'' = 0$, $L' - \lambda_3 L'' = 0$, $L' - \lambda_4 L'' = 0$, take either pair of *conjugates* as *base-rays*, say first and third, and express the other pair through them; thus, $L' - \lambda_1 L'' = L_1$, $L' - \lambda_3 L'' = L_3$; whence, finding L' and L'', and substituting, we find the other pair are

$$L_1-\frac{\lambda_1-\lambda_2}{\lambda_3-\lambda_2}L_3=0,\quad L_1-\frac{\lambda_1-\lambda_4}{\lambda_3-\lambda_4}L_3=0\,;$$

hence, $$\{1\ 2\ 3\ 4\}=\frac{\overline{\lambda_1-\lambda_2}\cdot\overline{\lambda_3-\lambda_4}}{\overline{\lambda_2-\lambda_3}\cdot\overline{\lambda_4-\lambda_1}}.$$

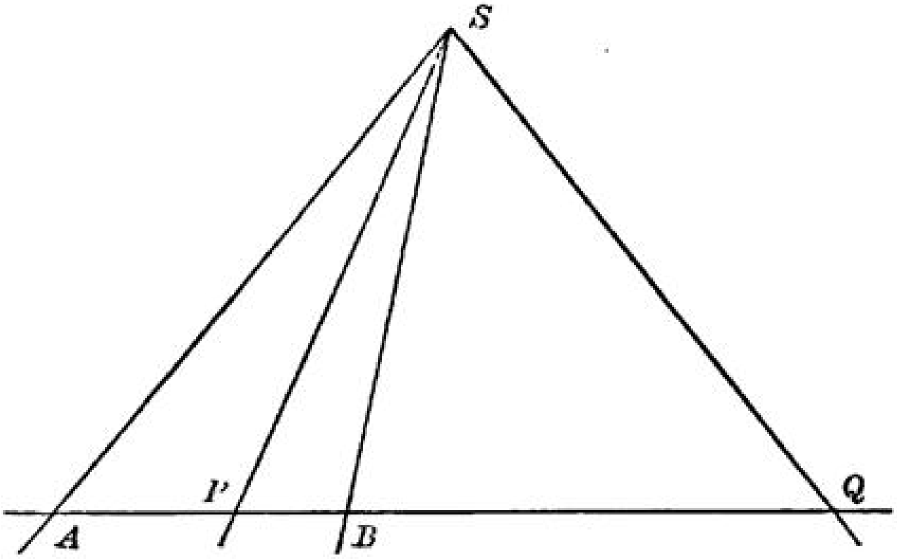
43. If $\lambda_2:\lambda_4=-1$, the pencil is harmonic; that is, specially, $L'=0$, $L''=0$, and $L'-\lambda L''=0$, $L'+\lambda L''=0$ are two harmonic pairs.

The four rays (1, 2, 3, 4) of the pencil (L', L'') are harmonic when, and only when,

$$\frac{\overline{\lambda_1-\lambda_2}\cdot\overline{\lambda_3-\lambda_4}}{\overline{\lambda_2-\lambda_3}\cdot\overline{\lambda_4-\lambda_1}}=-1\,;$$

i.e., when $$\lambda_1\lambda_3-\tfrac{1}{2}\cdot\overline{\lambda_1+\lambda_3}\cdot\overline{\lambda_2+\lambda_4}+\lambda_2\lambda_4=0.$$

44. If A, B and P, Q be two pairs of harmonic points, then for P midway between A and B, Q is at ∞; as P moves out from its mid-position toward B, Q moves out from its mid-position at ∞ into finity toward B. As P falls on B, so does



Q. The same remarks hold when A is put for B; also, when rays SA, etc., are put for points, it is necessary only to note that for Q in ∞ SQ is $\parallel$ to the carrier of the range.

If the CR. $=+1$, clearly a pair of conjugates (rays or points) must fall together.

45. If $L_1 = 0$, $L_2 = 0$ meet at S, and $L_1' = 0$, $L_2' = 0$ at S', any ray, $L_1 - \lambda L_2 = 0$, of the one pencil is said to *correspond* to the ray, $L_1' - \lambda L_2' = 0$, of the other; the pencils are called **homographic**. Since the CR. of two pairs of rays depends only on their parameters, the λ's, it follows that:

The CR. of two pairs of rays equals the CR. of the corresponding pairs in any other homographic pencil.

By eliminating λ we get $L_1 L_2' - L_2 L_1' = 0$; this, then, is a relation between the Cds. (x, y) of any junction-point of two corresponding rays; since the L's are of first degree in x and y, this Eq. is of second degree in x and y; hence,

The junction-points of pairs of corresponding rays in two homographic systems (pencils) lie on a curve of second degree.

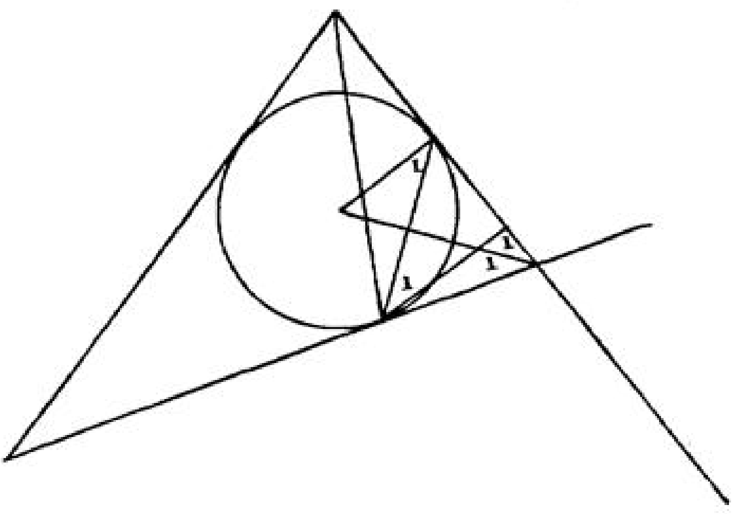
EXERCISES.

1. $N_1 = 0$ and $N_2 = 0$ enclose an $\measuredangle\ \alpha$; find the Eqs. of the $\perp$s to them through their intersection (N_1, N_2).

2. How do $N_1 - \lambda N_2 = 0$ and $\lambda N_1 - N_2 = 0$ lie in the pencil (N_1, N_2)?

3. Show that the transversals from the vertices of a $\triangle$ to the contact-points of the inscribed resp. escribed circles meet in a point.

4. $N_1 + \lambda_1 N_2 = 0$ and $N_1 + \lambda_2 N_2 = 0$ being taken for base-rays in $N_1 + \lambda N_2 = 0$, what is the value of h when $N_1 + \lambda' N_2 = 0$ is the same as $N_1 + \lambda_1 N_2 + h(N_1 + \lambda_2 N_2) = 0$?



5. Is $3x - 10y + 4 = 0$ of the pencil $5x - 7y + 3 + \lambda(2x + 3y - 1) = 0$? If so, express its Eq. through

$$7x - 4y + 2 = 0$$

and

$$19x + 14y - 4 = 0$$

as base-rays.

6. The CR. of a pencil is r; three rays are $L_1 = 0$, $L_2 = 0$, $L_1 + \lambda L_2 = 0$; what is the fourth? E.g., $11x - 2y + 7 = 0$, $3x + 5y = 6$, $17y = 2x + 25$, $r = 9:8$.

7. Four rays of the pencil $5x-7y+3+\lambda(2x+3y-1)=0$ are $11x+2y=0$, $x+5=13y$, $7x+2=4y$, $15x+8y=2$; find the CR. {First find $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.}

8. The inner halvers of the $\angle$s of a $\triangle$ are cut harmonically, each by the centres of an escribed and the inscribed circle.

9. Are these two pairs of rays harmonic:
$3y+4x=25$, $2y-3x=11$; $7y-2x=47$, $10x-y=3$?

10. Find the fourth harmonic to
$$L_1-\lambda L_2=0,\quad L_1+\lambda L_2=0,\quad L_1-\lambda' L_2=0.$$

11. Find the harmonic conjugates to each of the three rays in the pencil $L'+\lambda_1 L''=0$, $L'+\lambda_2 L''=0$, $L'+\lambda_3 L''=0$. E.g., $L'=2x+3y-5$, $L''=7x-2y+1$; take as rays, $13y-33x=10$, $23x-3y=2$, $9x+y=4$.

Involution.

46. To any one pair of conjugates there is an infinity of harmonic pairs of conjugates; for the Eq. that says the pair (λ_1, κ_1) is harmonic to the pair (λ, κ):

$$2\lambda\kappa-\overline{\lambda+\kappa}\cdot\overline{\lambda_1+\kappa_1}+2\lambda_1\kappa_1=0 \tag{1}$$

is clearly fulfilled for an ∞ of values of λ and κ. If a second pair (λ_2, κ_2) be also harmonic to the same pair (λ, κ), then must hold the second Eq.,

$$2\lambda\kappa-\overline{\lambda+\kappa}\cdot\overline{\lambda_2+\kappa_2}+2\lambda_2\kappa_2=0. \tag{2}$$

These two Eqs. are linear in $\lambda\kappa$ and $\lambda+\kappa$; hence they are both satisfied by one, and only one, pair of real values of $\lambda\kappa$ and $\lambda+\kappa$; this pair will yield one, and only one, pair of values of λ and κ, which may be real or imaginary; hence,

In any pencil there is always one, and only one, pair of rays, real or imaginary, harmonic to each of two given pairs.

47. If, now, there be a third pair (λ_3, κ_3) harmonic to the same pair (λ, κ), then must hold the third Eq.,

$$2\lambda\kappa-\overline{\lambda+\kappa}\cdot\overline{\lambda_3+\kappa_3}+2\lambda_3\kappa_3=0. \tag{3}$$

These three Eqs., (1), (2), (3), will be fulfilled by the same pair of values (λ, κ) when, and only when,

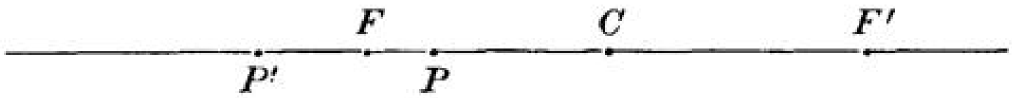
$$\begin{vmatrix} 1 & \lambda_1+\kappa_1 & \lambda_1\kappa_1 \\ 1 & \lambda_2+\kappa_2 & \lambda_2\kappa_2 \\ 1 & \lambda_3+\kappa_3 & \lambda_3\kappa_3 \end{vmatrix} = 0,$$

which is therefore the *Eq. of condition* declaring that the three pairs of rays, $L'-\lambda_1 L''=0$, $L'-\kappa_1 L''=0$; $L'-\lambda_2 L''=0$, $L'-\kappa_2 L''=0$; $L'-\lambda_3 L''=0$, $L'-\kappa_3 L''=0$, have a *common harmonic* pair.

Three or more pairs of rays harmonic each to the same pair form an **Involution.** The common harmonic we may call *focal rays.* Any transversal is cut by an *involution of rays* in an *involution of points*, whose *foci* are the section-points of the *focal rays.*

Pairs of conjugate points correspond to pairs of conjugate rays. The foci and a pair of conjugate points form an harmonic range. The mid-point between the foci is called **Centre** of the Involution.

48. *The product of the central distances of a pair of conjugate points is a constant:* the squared half-distance between the foci.



For $\{FPF'P'\} = FP\cdot F'P' : PF'\cdot P'F = -1$; if $FF'=2c$, $CP=d$, $CP'=d'$, then $(c+d)\cdot(d'-c) = -(c+d')\cdot(d-c)$, or $dd'=c^2$.

If P fall on F or F', so must P'; i.e., foci are *double* points, and focal rays are *double* rays. If P and P' fall on the same side of C, d and d' are like-signed, c^2 is $+$, c is real, the foci and focal rays are real; but if P and P' fall not on the same side of C, d and d' are unlike-signed, c^2 is $-$, c is imaginary, the foci and focal rays are imaginary.

The foci fix the centre, and so the Involution; but, by Art. 46, two pairs of conjugates fix the common harmonic pair; i.e., the focal rays resp. points; hence, *two pairs of conjugates* (rays resp. points) *fix an Involution.*

49. If $L'=0$, $L''=0$ be focal rays, $L'-\lambda_s L''=0$, where $s=|1, 2, 3, 4|$, any four rays, then their conjugates are $L'+\lambda_s L''=0$ (Art. 43), and the CRs. of the two sets are plainly equal; hence,

In any Involution the CRs. of any four elements and of their conjugates are equal.

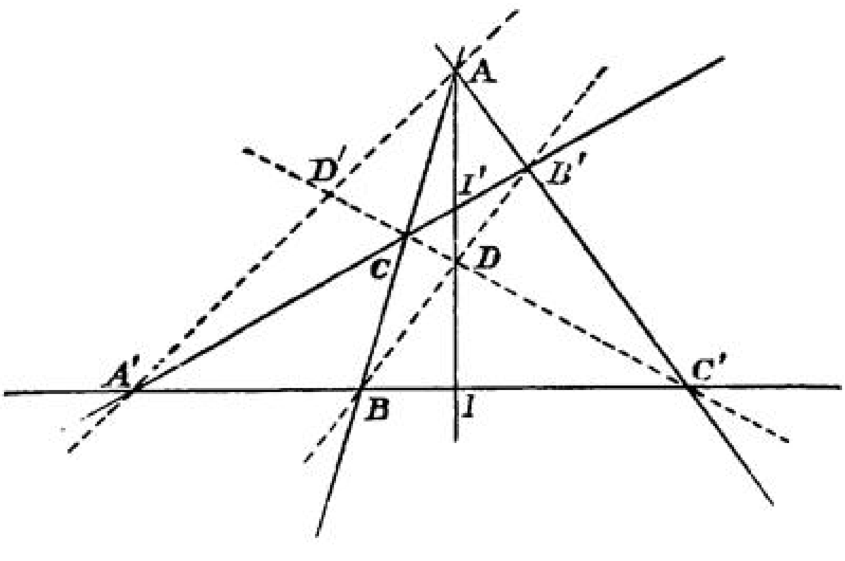
COR. If six points (or rays), A, A', B, B', C, C', be in involution, then

$$\{ABCA'\}=\{A'B'C'A\},$$

which tests whether the third pair be involved with the other two.

Cross Ratio and Involution are of greatest import to Higher Geometry and Mechanics. Minuter treatment were out of place in this elementary work; but the foregoing, it is hoped, may excite the reader's interest, and incite him to further pursuit of the subject. A single illustration is added in the proof of the familiar theorem:

Each diagonal of a four-side is cut harmonically by the other two.



Be AB, AC', $A'B$, $A'C$ the sides, AA', BB', CC', the diagonals. Then

$$\begin{aligned}\{CDC'D'\} &= A\{CDC'D'\}=\{CI'B'A'\}\\ &= D\{CI'B'A'\}=\{C'IBA'\}\\ &= A\{C'IBA'\}=\{C'DCD'\}\\ &= 1:\{CDC'D'\};\ \therefore \{CDC'D'\}=\pm 1.\end{aligned}$$

But a CR. can $=1$ only when two rays fall together; hence, Q.E.D. Conduct the proof for the other diagonals.

Equations of Higher Degree representing Several Right Lines.

50. If $L_1=0,\ L_2=0,\ \cdots,\ L_n=0$ be Eqs. of n RLs., their product $L_1\cdot L_2\cdots L_n=0$ will be of nth degree in x and y, and will be satisfied by all and only such pairs of values of x and y as reduce some factor, as L_r, to 0; i.e., the Eq. $L_1 L_2\cdots L_n=0$ will picture all and only such points as lie on the RLs. $L_1=0,\ L_2=0,\ \cdots,\ L_n=0$.

If all the L's be homogeneous in x and y, so will be their product, and not otherwise; but then all the RLs. go through the origin; $\therefore$ *n RLs. through the origin are pictured by an Eq. of nth degree, and homogeneous in x and y.*

Conversely, *such an Eq. pictures n RLs. through the origin.*

For, on division by x^n, the Eq. takes the form

$$c_0+c_1\frac{y}{x}+c_2\frac{y^2}{x^2}+\cdots+c_k\frac{y^k}{x^k}+\cdots+c_n\frac{y_n}{x_n}=0.$$

This Eq. of nth degree in the ratio $y:x$ has n roots, $s_1,\ s_2,\ s_3,\ \ldots,\ s_n$, and may be written

$$c_n\left\{\frac{y}{x}-s_1\right\}\cdot\left\{\frac{y}{x}-s_2\right\}\cdots\left\{\frac{y}{x}-s_k\right\}\cdots\left\{\frac{y}{x}-s_n\right\}=0.$$

This Eq. is satisfied when, and only when, a factor equals 0: as $\dfrac{y}{x}-s_k=0$, or $y=s_k x$. But this is the Eq. of a RL. through the origin, and there are n such factors.

The RLs. are real or imaginary, separate or coincident, according as the roots, the s's, are real or imaginary, unequal or equal.

51. If the Eq. be not homogeneous in x and y, we may test whether it be resoluble into factors of first degree in x and y

by assuming such a resolution, forming the product of the assumed factors, and equating coefficients of like terms in x and y in the assumed and given expressions. For an Eq. of nth degree there must then hold $\frac{n\cdot\overline{n-1}}{1\cdot 2}$ Eqs. of condition among its coefficients. Our immediate concern is only with the Eq. of second degree, which may be written thus:

$$kx^2+2hxy+jy^2+2gx+2fy+c=0. \qquad (1)$$

Instead of the tedious general method we may employ the following, especially as its incidental results are useful:

Pass to || axes through the section-point (x_1, y_1) of the two RLs. which the above Eq. is supposed to represent. This is done by putting $x+x_1$ for x, $y+y_1$ for y; on collection there results

$$kx^2+2hxy+jy^2+2g'x+2f'y+c'=0, \qquad (2)$$

where $\quad g'=kx_1+hy_1+g, \qquad f'=hx_1+jy_1+f,$

$$c'=kx_1^2+2hx_1y_1+jy_1^2+2gx_1+2fy_1+c.$$

This result is got by reasoning thus: terms not containing x_1 or y_1 are found by supposing $x_1=0$, $y_1=0$, which gives the original expression; terms not containing x or y, by supposing $x=0$, $y=0$, which gives the original expression with x_1, y_1 written for x, y; terms containing a subscribed and an unsubscribed letter can result only from the original terms of second degree, appear doubly, and are symmetric as to the subscribed and unsubscribed letters. This reasoning gives the following as the result of the substitution:

$$kx^2+2hxy+jy^2+2gx+2fy+kx_1^2+2hx_1y_1+jy_1^2+2gx_1$$
$$+2fy_1+2kxx_1+2h(x_1y+y_1x)+2jyy_1+c=0;$$

and this collected gives Eq. (2). It is important that the student thoroughly master this argument.

Now, if Eq. (1) is a pair of RLs. through (x_1, y_1), Eq. (2) is a pair through the origin; hence, (2) must be homogeneous of second degree in x and y; hence, terms of lower degree must vanish; hence, $g'=0$, $f'=0$, $c'=0$. But c' may be written

$$(kx_1+hy_1+g)x_1+(hx_1+jy_1+f)y_1+(gx_1+fy_1+c)=0;$$

and the coefficients of x_1 and y_1, being g' and f', are 0; hence, so is gx_1+fy_1+c; hence, between x_1 and y_1 must consist the three Eqs.,

$$kx_1+hy_1+g=0,\quad hx_1+jy_1+f=0,\quad gx_1+fy_1+c=0.$$

The condition of this consistence is

$$\begin{vmatrix} k & h & g \\ h & j & f \\ g & f & c \end{vmatrix}=0.$$

This Determinant is named *Discriminant* of the Eq. of second degree, and is denoted by Δ. Hence, $\Delta=0$ *is the condition that the Eq. of second degree represent two RLs.*

The co-factor of any element of Δ will be denoted by the corresponding capital letter. The Cds. (x_1, y_1) of the section of the RLs. may be found from any two of the above three Eqs., and are $x_1=G:C$, $y_1=F:C$. If C, or $kj-h^2$, be $\gtrless 0$, x_1 and y_1 are finite, the RLs. meet in finity; if $C=0$, x_1 and y_1 are ∞, the RLs. meet in ∞; i.e., the RLs. are ∥.

52. It is to note that changing the origin but not the axial directions does not change the terms of highest, i.e., of second, degree.

To find the direction coefficients, it suffices to factor

$$kx^2+2\,hxy+jy^2,$$

or

$$y^2+2\frac{h}{j}\,xy+\frac{k}{j}\,y^2.$$

If the factors be $y-s_1x$, $y-s_2x$,

then

$$s_1+s_2=-2\frac{h}{j},\quad s_1s_2=\frac{k}{j};$$

whence,

$$s_1-s_2=2\frac{\sqrt{h^2-kj}}{j},$$

$$s_1 = \{-h + \sqrt{h^2 - kj}\} : j,$$

$$s_2 = \{-h - \sqrt{h^2 - kj}\} : j.$$

Hence, by Art. 26, if the RLs. enclose an $\measuredangle\ \phi$,

$$\tan\phi = 2\sqrt{h^2 - kj}\cdot\sin\omega : \{k + j - 2h\cos\omega\},$$

or $$\tan\phi = 2\sqrt{h^2 - kj} : \{k + j\};$$

the RLs. are $\perp$ when

$$k + j - 2h\cos\omega = 0, \quad \text{or when} \quad k + j = 0, \ \text{if} \quad \omega = 90^\circ,$$

and are imaginary when $h^2 - kj < 0$.

53. *To find the pair of RLs. halving the $\measuredangle$s between the pair*

$$kx^2 + 2hxy + jy^2 = 0, \ \text{or} \ y - s_1x = 0 \ \text{and} \ y - s_2x = 0.$$

Brought to the N.F., these Eqs. are

$$\overline{y - s_1x} : \sqrt{1 + s_1^2} = 0, \qquad \overline{y - s_2x} : \sqrt{1 + s_2^2} = 0.$$

Their sum resp. difference is the Eq. of the outer resp. inner bisector; and the product of these is

$$\overline{y - s_1x}^2 : \overline{1 + s_1^2} - \overline{y - s_2x}^2 : \overline{1 + s_2^2} = 0,$$

or $$\overline{s_1^2 - s_2^2}\cdot y^2 - 2\cdot\overline{s_1s_2 - 1}\cdot\overline{s_1 - s_2}\cdot xy - \overline{s_1^2 - s_2^2}\cdot x^2 = 0.$$

Divide by $\overline{s_1 - s_2}$, replace $s_1 + s_2$, s_1s_2 out of Art. 52; whence,

$$y^2 + \frac{k - j}{h}\,xy - x^2 = 0, \quad \text{the Eq. sought.}$$

Note that this pair of halvers are *always real*, though the pair $kx^2 + 2hxy + jy^2 = 0$, whose $\measuredangle$s they halve, may be imaginary.

54. The general Eq. of second degree is *not* homogeneous in x and y; we may make it so by multiplying the terms of lower degree by fit powers of some linear function of x and y. So we get

$$kx^2 + 2hxy + jy^2 + 2gx + 2fy + c = 0, \qquad (A)$$

$$lx + my = 1, \qquad (B)$$

$$kx^2 + 2hxy + jy^2 + (2gx + 2fy)(lx + my) + c(lx + my)^2 = 0. \qquad (C)$$

By Art. 50, (C) represents two RLs. through the origin, and by Art. 30, they go through the intersections of (A) and (B).

Note the method of this article, as it solves the problem of finding the Eq. of the RLs. through the origin and the section-points of a RL. with a curve of any degree.

EXERCISES.

1. What RLs. are pictured by the Eqs.:

(a) $\overline{x-c}\,.\,\overline{x-d} = 0$;
(b) $xy' = 0$;
(c) $x^2 - 4y^2 = 0$;
(d) $x^2 - y^2 = 0$;
(e) $x^2 - 5xy + 4y^2 = 0$;
(f) $4y^2 - 15xy - 4x^2 = 0$;
(g) $y^2 + 13xy - 10x^2 = 0$;
(h) $y^2 - 2xy + 3x^2 = 0$;
(i) $y^2 - 2xy\sec\theta + x^2 = 0$;
(j) $x^2 + xy - 6y^2 + 7x + 31y - 18 = 0$?

2. Find the ∡s of the above pairs, and their bisectors.

3. For what values of λ do these Eqs. picture RLs.:

(a) $12x^2 - 10xy + 2y^2 + 11x - 5y + \lambda = 0$;
(b) $12x^2 + \lambda xy + 2y^2 + 11 - 5y + 2 = 0$;
(c) $12x^2 + 36xy + \lambda y^2 + 6x + 6y + 3 = 0$?

4. Show that the RLs. joining the origin to the section-points of $3x^2 + 5xy - 3y^2 + 2x + 3y = 0$ and $3x = 2y + 1$ are ⊥.

5. $N_1 = 0$, $N_2 = 0$, $N_3 = 0$ being sides of a △, find the RLs.:

$N_1 + N_2 + N_3 = 0$; $\quad N_1 - N_2 + N_3 = 0$;
$N_1 + N_2 - N_3 = 0$; $\quad -N_1 + N_2 + N_3 = 0$.

Rectilinear Loci.

55. *The sum total of positions to which a point is astricted by some geometric condition is called the* **locus** *of the point.*

To express the condition through constants and the current Cds. of the point is to find the Eq. of the locus.

For doing this no fixed rule can be given; each problem or class of problems must be solved for itself.

In general, the expressions will be made more simple by choosing the axes with special reference to the figure of the problem, but more symmetric by avoiding such reference; sometimes the one advantage, sometimes the other, will outweigh. Often the point is fixed as the junction of pairs of corresponding lines in two systems of lines; the *common parameter* of the two systems must then be eliminated. Thus, if the point lie on the curve $F(x, y; p) = 0$, and also on the curve $\phi(x, y; p) = 0$, by giving p any value we may find a pair of values of x and y satisfying both Eqs.; i.e., we may find *one* position of the point; but by eliminating p we find a relation between *every* pair (x, y) which satisfies the two Eqs., whatever p may be; i.e., we find a relation between the Cds. of the point in *any* position; i.e., we find the Eq. of its locus. There may be more than one parameter; the number of Eqs. needed is, in general, one greater than the number of parameters.

EXERCISES.

1. A point moves so that the sum of its distances from the sides of an $\measuredangle$ is constant; what is the point's path?

If $l_1x + m_1y + n_1 = 0$, $l_2x + m_2y + n_2 = 0$ be the sides of the $\measuredangle$, then
$\frac{l_1x + m_1y + n_1}{\sqrt{l_1^2 + m_1^2}} + \frac{l_2x + m_2y + n_2}{\sqrt{l_2^2 + m_2^2}} = s$ is the locus, a RL.

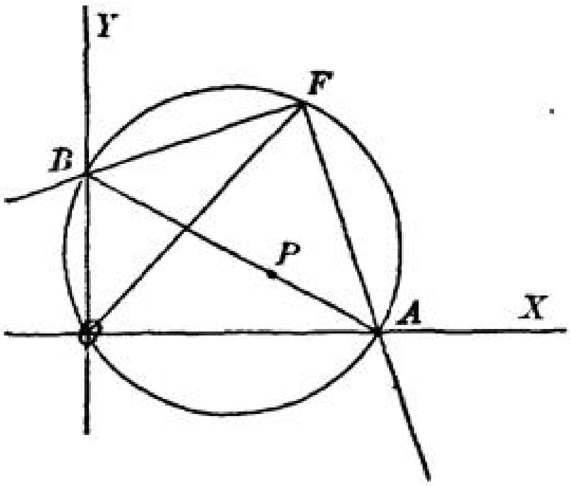
The sides being axes, at once $x + y = s : \sin\omega$. Draw the RL.

2. The sum of a point's distances from n RLs. is s; find its path.

3. The ratio of a point's distances from two RLs. is $\mu_1 : \mu_2$; find its path.

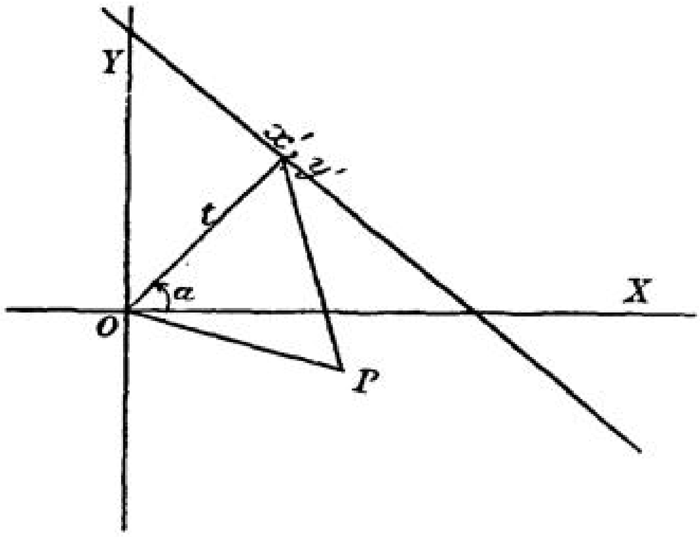
4. From side to side of a given RL. are drawn ∥ tracts; a point cuts them all in a fixed ratio; find its path.

5. The ends of the hypotenuse of a right △ move on the rectang. Cd. axes; where does the vertex move?



6. Given one R. ∡ fixed, another turning about a fixed point F; find the locus of P cutting the junction-line of the sections of fixed and moving sides, in the ratio $\mu_1 : \mu_2$.

Hint. Take the fixed sides as axes, express OA resp. OB through x resp. y, and project on OF.



7. Tracts are drawn from the origin to any point of a RL., $y = sx + b$; find the locus of the vertices of equilateral △s constructed on the tracts.

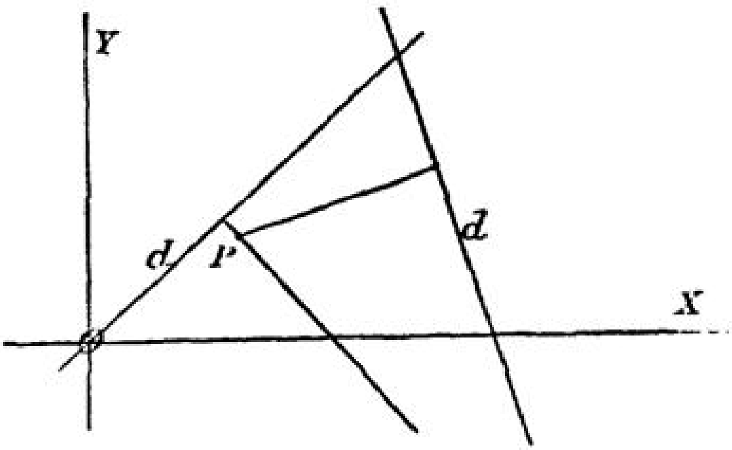
Take the slope α of a tract to the X-axis as parameter, and proceed thus:

$$y' = b : (1 - s \cot \alpha),$$
$$t = y' : \sin \alpha$$
$$= b : (\sin \alpha - s \cos \alpha);$$

$$x = t \,.\, \cos(60^\circ - \alpha) = b \cos(60^\circ - \alpha) : (\sin \alpha - s \cos \alpha)$$
$$= b\,(1 + \sqrt{3} \,.\, \tan \alpha) : 2\,(\tan \alpha - s);$$
$$y : x = -\tan(60^\circ - \alpha) = (\tan \alpha - \sqrt{3}) : (1 + \sqrt{3} \,.\, \tan \alpha);$$
$$\therefore \tan \alpha = (x\sqrt{3} + y) : (x - y\sqrt{3});$$

whence $(y + s\sqrt{3}) + x\,(\sqrt{3} - s) = 2b,$

a RL. sloped 60° to the given RL. **Draw it.**



8. Find the locus of the intersection of ⊥s to the sides of a △, cutting the sides at points equidistant from the ends of the base.

Take the base as X-axis, either end as origin; the Eqs. of the sides are $y = s_1 x$, $y = s_2 x + b$; take the distance d as parameter; the

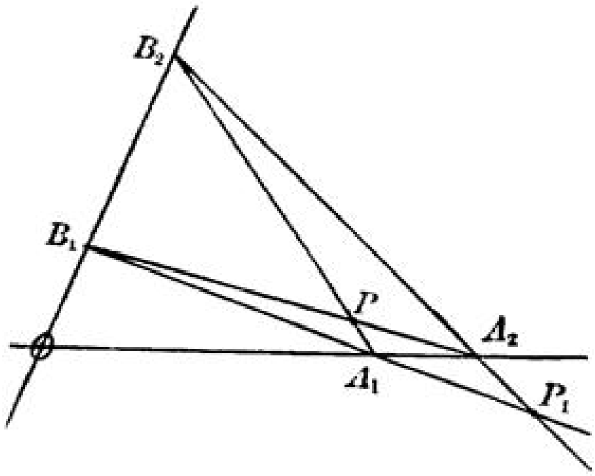
Eqs. of the ⊥s are $y = -\frac{1}{s_1}x + c_1$, $y = -\frac{1}{s_2}x + c_2$; express c_1, c_2 through d, so we find the ⊥s are $s_1y + x = d\sqrt{1 + s_1^2}$, $s_2y + x + \frac{b}{s_2} = -d\sqrt{1 + s_2^2}$. Hence eliminate d.

9. Find the locus of the intersection of ⊥s to the sides of a △ through the points where they caut ∥ to the base.

10. On the sides of a given ∡ are laid off from its vertex tracts whose sum is s; ⊥s to the sides through the ends of the tracts meet at P. Where is P?

11. Given a vertex, the directions of the sides, and the sum of the sides, of a parallelogram; find the locus of the opposite vertex.

12. Find the locus of the intersection of RLs. joining crosswise the points where pairs of rays through a fixed point cut the sides of a fixed ∡.



If the axial intercepts of the rays be a_1, b_1, a_2, b_2, the cross lines are

$$\frac{x}{a_1} + \frac{y}{b_2} = 1, \qquad \frac{x}{a_2} + \frac{y}{b_1} = 1;$$

if (x_1, y_1) be the fixed point,

$$\frac{x_1}{a_1} + \frac{y_1}{b_1} = 1, \qquad \frac{x_1}{a_2} + \frac{y_1}{b_2} = 1.$$

Form the difference of the first, and also of the second, pair of Eqs.; their quotient gives $\frac{x}{y} = -\frac{y_1}{x_1}$, the locus sought, which is seen to be the same for all positions of P_1 on the RL. OP_1.

In the following problems, about a △ ABC, take AB as the $+X$-axis of rectang. Cds.

Five noteworthy points of a △ are: *mass-centre* (intersection of medials), *orthocentre* (intersection of altitudes), *centre of vertices* (or of circumscribed circle), *centre of sides* (or of inscribed circle), *intersection of transversals* from vertices to contact-points of opposite escribed circles.

13. Find the loci of these points for similar △s with a fixed ∡ A. If r be the radius of the circumscribed circle, the sides of the △ are $2r.\sin A$, $2r.\sin B$, $2r.\sin C$; the Cds. of the points (in order) are:

$x = \frac{2}{3} r (\sin A \cos B + 2 \cos A \sin B)$, $y = \frac{2}{3} r \sin A . \sin B$;

$x = 2r \cos A \sin B$, $y = 2r \cos A \cos B$;

$x = r \sin \overline{A + B}$, $y = -r \cos \overline{A + B}$;

$x = 4r \cos \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \cos \frac{A+B}{2}$, $y = 4r \sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \cos \frac{A+B}{2}$;

$x = 2r (\sin A - \sin B + \cos A . \sin B)$, $y = 8r \overline{\sin \frac{A}{2}}^2 \cdot \overline{\sin \frac{B}{2}}^2$.

Eliminate the parameter r from each pair; it is seen that as the △ swells, the five points move out on five RLs. through the centre of similitude.

14. Find locus of centre of sides, only A and its counter-side a constant.

15. Find locus of orthocentre, only B and b constant.

16. Find loci of first, second, third, fifth points, only B and a constant.

17. Find loci of the first four points, only A and b constant.

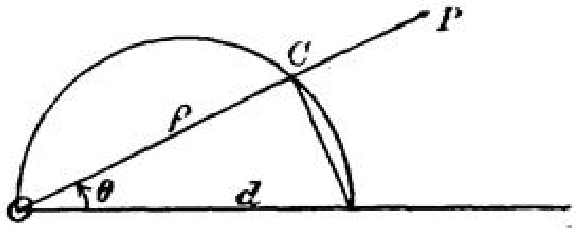
18. Find loci of the first four points, only A and c constant.

19. Find loci of third and fourth points, only C and b constant.

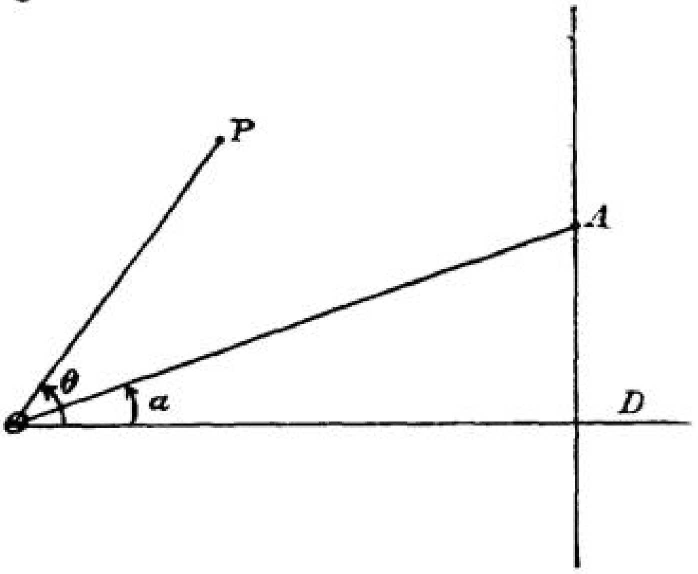
20. Find locus of mass-centre when c is constant in size and position, while C moves on the RL. $y = sx + n$.

21. Given an ∡ of a △ and the sum of the including sides; find locus of P cutting third side in a fixed ratio.

In problems about tracts of changing length and direction, measured from a fixed point, polar Cds. are recommended.



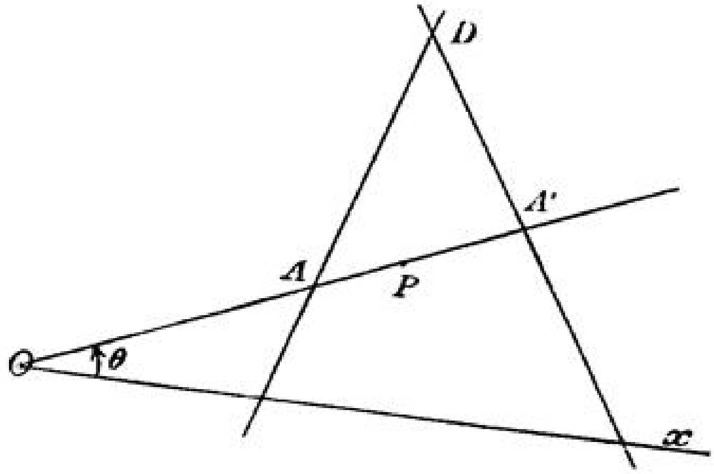
22. Chords through a fixed point of a circle are produced till the rectangle of chord and chord produced is constant; find locus of end of produced chord. Take the diameter through the fixed point as polar axis; then $OP = \rho$, $OC = d \cos \theta$, $OC . OP = k^2$; $\rho \cos \theta = k^2 : d$, a RL., as is also clear from similar ⚠.



23. Two tracts whose lengths are in a fixed ratio enclose a fixed ∡ at a fixed point, and the end of one moves on a RL.; how moves the end of the other?

Take as polar axis the ⊥ from the fixed point on the fixed RL.; be OA and OP the tracts, and $OP : OA = r$. Then $\rho = rp \sec(\theta - a)$, a RL. Draw it.

24. From a fixed point O is drawn a ray cutting two fixed RLs. at A and A', and on it P is taken so that OP is the harmonic mean of OA and OA'; find locus of P.



OX being taken as polar axis, the Eqs. of the fixed RLs. DA and DA' are

$$1 : \rho = l \cos\theta + m \sin\theta,$$
$$1 : \rho = l' \cos\theta + m' \sin\theta;$$

these Eqs. hold for the same θ where ρ is OA resp. OA' in the first resp. second. By hypothesis, $2 : OP = 1 : OA + 1 : OA'$. Writing ρ for OP, we get $2 : \rho = (l + l')\cos\theta + (m + m')\sin\theta$, the Eq. of a RL. through D.

25. Generalize Ex. 24 by taking n instead of 2 RLs. through D.

26. Given base and difference of sides of a △; at either end of base is drawn a ⊥ to the conterminous side; find locus of its intersection with the inner halver of the vertical angle.

Families of Right Lines through a Point.

56. Thus far in each problem have been two conditions, enabling us to determine the two parameters in the general Eq. of a RL. Had there been but one condition in the problem, the result would have contained one parameter undetermined, and so would have represented a family of RLs. Should a parameter appear in a result linearly, then, by Art. 29, all RLs. of the family would pass through a point. When the parameters appear linearly, both in the general Eq. of a RL. and in some Eq. of condition, all RLs. of the family go through a point; for, by help of the Eq. of condition, we may eliminate one parameter and leave the other appearing linearly.

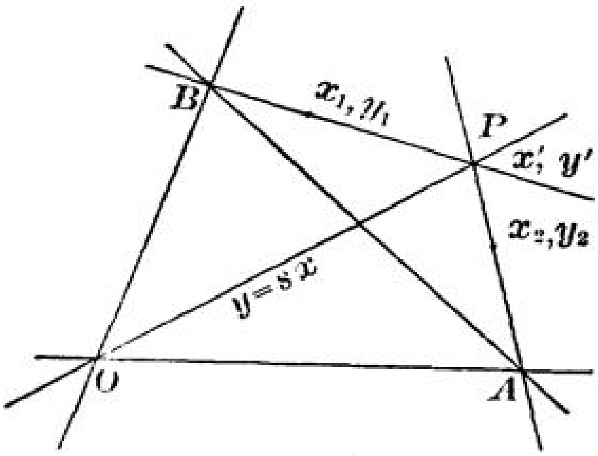
Such is the case, e.g., when the parameters are the current Cds. x', y' of a RL.; for then they fulfil some linear condition, $lx' + my' + n = 0$.

EXERCISES.

1. The vertical ∡ and the sum of the reciprocals of its sides are given in a △; find the Eq. of the base.

Take the sides as axes; then, the reciprocals of intercepts of the base on the axes being l and m, the Eq. of the base is $lx + my = 1$; also, $l + m = c$. Hence the base turns about a fixed point. Find the point.

2. If, as the vertices of a △ move on three fixed RLs. through a point, two sides turn about fixed points, the third turns about a fixed point.



Take two of the fixed RLs., as OA, OB, for axes; then $y = sx$ is the third RL. OC. Be (x_1, y_1), (x_2, y_2), the fixed points, (x', y') the third vertex C; then $y' = sx'$, and the Eqs. of AC and BC are

$$(x_1 - x')\, y - (y_1 - sx')\, x + x'\,(y_1 - sx_1) = 0,$$
$$(x_2 - x')\, y - (y_2 - sx')\, x + x'\,(y_2 - sx_2) = 0.$$

Hence, find OA, OB, and form the Eq. of the third side AB,

$$x\,(y_2 - sx') : (y_2 - sx_2) - y\,(x_1 - x') : (y_1 - sx_1) = x'.$$

The parameter x' enters this Eq. linearly; hence AB turns about a a point. Find the point as the intersection of two base RLs. by solving the Eq. as to x'.

3. All RLs., the sum of proper multiples of the distances of n fixed points from any one of which equals 0, form a family through a point (the *centre of proportional distances of the points*).

Be (x_k, y_k) one of the points, μ_k the proper multiplier, $x \cos\alpha + y \sin\alpha - p = 0$ one of the RLs. Then, by hypothesis,

$$\sum_{k=1}^{k=n} \mu_k x_k \cos\alpha + \sum_{k=1}^{k=n} \mu_k y_k \sin\alpha - p \sum_{k=1}^{k=n} \mu_k = 0.$$

Between this Eq. and the Eq. of the RL. eliminate p; on division by $\cos\alpha$ (which, and $\sin\alpha$, may be written before the summation sign Σ) $\tan\alpha$ appears, as parameter in the result, linearly. By proceeding as directed in Ex. 2, the fixed point is found to be

$$(\Sigma \mu_k x_k : \Sigma \mu_k, \quad \Sigma \mu_k y_k : \Sigma \mu_k).$$

CHAPTER III.

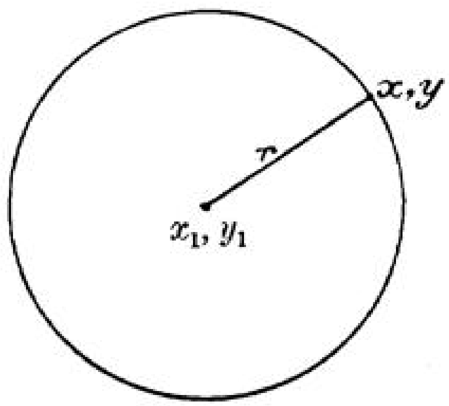
THE CIRCLE.

Before treating the general Eq. of second degree, it may be well to treat a special case of great importance.

57. By Art. 15, if r be the distance between (x, y) and (x_1, y_1), then

$$\overline{x-x_1}^2+\overline{y-y_1}^2+2\cdot\overline{x-x_1}\cdot\overline{y-y_1}\cdot\cos\omega=r^2. \qquad (\text{I}')$$

Hold r and (x_1, y_1) fast, letting (x, y) vary; (x, y) will keep r distant from (x_1, y_1); the sum total of its positions is called a **Circle**, with radius r, centre (x_1, y_1). Hence (I′) is the rectilinear Eq. of a circle.



For $\omega = 90°$ the important rectang. Eq. is

$$\overline{x-x_1}^2+\overline{y-y_1}^2=r^2. \qquad (\text{I})$$

For $x_1 = 0$, $y_1 = 0$, i.e., for the central Eq., the simpler forms are

$$x^2+y^2+2xy\cos\omega=r^2, \qquad (\text{II}')$$

and $$x^2+y^2=r^2. \qquad (\text{II})$$

58. A circle is known completely when are known its radius (in length) and its centre (in position). Since in (I′) resp. (I) radius and centre are expressed generally, it is clear that (I′) resp. (I) is a general form to which any Eq. in oblique resp. rectang. Cds. of any circle may be brought. We note that the coefficients of x^2 and y^2 are equal: each $=1$, and the coefficient of xy is $2\cos\omega$ resp. 0. Divided by k the general Eq. of second degree takes the form

$$x^2+2\frac{h}{k}xy+\frac{j}{k}y^2+2\frac{g}{k}+2\frac{f}{k}+\frac{c}{k}=0. \qquad \text{(III}')$$

Accordingly, *if this be the Eq. of a circle*, then must $j:k=1$ or $j=k$, and $h:k=\cos\omega$ or $h=k\cos\omega$.

These, then, are *conditions necessary* that the Eq. of second degree picture a circle. They are also *sufficient*, for where they are fulfilled, the three arbitraries x_1, y_1, r, may be so chosen as to satisfy any set of values of the three coefficients,

$$\frac{2g}{k},\quad \frac{2f}{k},\quad \frac{c}{k}.$$

Two problems may now be solved:

1. *Given centre and radius, to form the Eq. of the circle.*

Substitute in (I′) resp. (I), and reduce.

2. *Given the Eq. of the circle, to find centre and radius.*

Divide the Eq. by the coefficient of x^2, equate the coefficients of x and y, and the absolute term, to their correspondents in (I′) resp. (I), and so find x_1, y_1, r.

EXERCISES.

1. The centre of a circle is $(3,-4)$, radius 6; find its Eq.

2. Find the Eqs. of the circles whose centres and radii are $(0,0)$, 9; $(7,0)$, 3; $(0,-2)$, 11; $(-4,17)$, 1.

3. Can these Eqs. picture circles, and of what radii and centres:

$$15x^2+15y^2+15xy-90y-45=0;$$
$$3x^2-3xy+3y^2-9x+12y-10=0\ ?$$

4. Find centres and radii of $7x^2-7y^2+49x+84y+14=0$ and $5x^2+5y^2+25x-15y+20=0$.

59. As is clear on comparing (I′) and (III′), the general Eqs. for determining x_1, y_1, r, are

$$x_1 + y_1 \cos\omega = -g : k,$$
$$y_1 + x_1 \cos\omega = -f : k,$$
$$x_1^2 + y_1^2 + 2x_1 y_1 \cos\omega - r^2 = c : k;$$

or, for rectang. axes, more simply,

$$x_1 = -g : k,$$
$$y_1 = -f : k,$$
$$x_1^2 + y_1^2 - r^2 = c : k;$$
$$\therefore\ r^2 = (g^2 + f^2 - kc) : k^2.$$

The equivalent forms, in which the coefficient of $x^2 + y^2$ is 1,

$$\overline{x - x_1}^2 + \overline{y - y_1}^2 - r^2 = 0,$$

or $$x^2 + y^2 + 2gx + 2fy + c = 0, \qquad \text{(III)}$$

may be called the Normal Form (N.F.) of the rectang. Eq.

In the N.F. $x_1 = -g$, $y_1 = -f$, $r^2 = g^2 + f^2 - c$; i.e., *the Cds. of the centre of a circle are the negative half-coefficients of x and y in the N.F. of its rectang. Eq.;* the squared radius is the sum of their squares less the absolute term.

The circle is a *real*, a *point-*, or an *imaginary*, circle, according as

$$g^2 + f^2 - kc > 0, \text{ or } = 0, \text{ or } < 0.$$

The Cds. of the centre do not contain the absolute term; hence, if this alone change, the centre does not change; i.e., *circles whose Eqs. differ only in absolute terms are concentric.*

If the absolute term be 0, the curve goes through the origin.

60. The Eq. of a circle contains three arbitraries, $x_1\ y_1\ r$, or g, f, c. Hence, three conditions are needed and enough to fix a circle. To find a circle fixed by three conditions, express these through Eqs. and thence find the arbitraries. Thus, find the circle through the points (1, 2), (3, −5), (−2, 1). Since each Cd. pair satisfies the general Eq.

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

we get
$$2g+4f+c+5=0,$$
$$6g-10f+c+34=0,$$
$$-4g+2f+c+5=0;$$

whence, finding g, f, c, and substituting, we get

$$23(x^2+y^2)-29x+87y-520=0.$$

Or, by Determinants, more neatly, thus: If the circle

$$x^2+y^2+2gx+2fy+c=0$$

goes through (x_1, y_1), (x_2, y_2), (x_3, y_3), then

$$x_k^2+y_k^2+2gx_k+2fy_k+c=0$$

for $k=1, 2, 3$. These four Eqs. consist for the same values of g, f, c when, and only when,

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix}=0,$$

which is therefore the Eq. sought. Clearly, it is also the condition that four points, P, P_1, P_2, P_3, lie on a circle.

EXERCISES.

1. Find the circles through $(0,0)$, $(a,0)$, $(0,b)$; $(a,0)$, $(-a,0)$, $(0,b)$.

2. If (x_1, y_1) (x_2, y_2) be ends of a diameter, the Eq. of the circle is

$$\overline{x-x_1}\cdot\overline{x-x_2}+\overline{y-y_1}\cdot\overline{y-y_2}=0,$$

or, for oblique axes,

$$\overline{x-x_1}\cdot\overline{x-x_2}+\overline{y-y_1}\cdot\overline{y-y_2}+\{\overline{x-x_1}\cdot\overline{y-y_2}+\overline{x-x_2}\cdot\overline{y-y_1}\}\cos\omega=0.$$

N.B. The following familiar propositions in the Theory of the Quadratic Equation are here recalled once for all:

Be $C_2x^2+2C_1x+C_0=0$ the general Eq. of second degree in x, r_1 and r_2 its roots; then

(1) $r_1 + r_2 = -2C_1 : C_2,\ \ r_1 \cdot r_2 = C_0 : C_2.$

(2) For $C_0 = 0$, one root $= 0$; for $C_0 = 0$ and $C_1 = 0$, both roots $= 0$.

(3) For $C_2 = 0$, one root $= \infty$; for $C_2 = 0$ and $C_1 = 0$, both roots $= \infty$.

(4) For $C_1 = 0$, $r_1 = -r_2$; i.e., the roots are equal, but unlike-signed.

(5) For $C_1^2 = C_0 C_2$, the roots are equal and like-signed.

61. To find where the axes cut a curve of second degree, equate y resp. x to 0 in the general Eq.; so we get

$$kx^2 + 2gx + c = 0 \quad \text{resp.} \quad jy^2 + 2fy + c = 0.$$

The roots of these Eqs. are the intercepts on the X- resp. Y-axis. (They are equal, i.e., the X- resp. Y-axis meets the curve in two consecutive points, i.e., is tangent to the curve when, and only when, $g^2 = kc$ resp. $f^2 = kc$. See Art. 64.)

Conversely, given the intercepts on the axes, to find the Eq. of the circle. Suppose $k = j = 1$, and be a_1, a_2 resp. b_1, b_2 the intercepts on the X- resp. Y-axis; then

$$2g = -(a_1 + a_2),$$
$$2f = -(b_1 + b_2),$$
$$c = a_1 a_2 = b_1 b_2.$$

Let the student interpret these Eqs. geometrically.

Hence, $x^2 + y^2 - (a_1 + a_2)x + 2fy + a_1 a_2 = 0,$

$$x^2 + y^2 + 2gx - (b_1 + b_2)y + b_1 b_2 = 0,$$

resp. $x^2 + y^2 - (a_1 + a_2)x - (b_1 + b_2)y + \frac{1}{2}(a_1 a_2 + b_1 b_2) = 0$

are equations of a circle in terms of its intercepts: on the X-axis, on the Y-axis, resp. on both axes.

N.B. Of course, only three intercepts can be assumed at will; then the fourth follows from $a_1 a_2 = b_1 b_2$.

EXERCISES.

1. Where do the axes cut $x^2+y^2+5x-y-6=0$?

2. A circle touches each axis a from the origin; find it.

3. Find the equation of a circle through $(0,2)$, $(0,-4)$, $(5,0)$.

4. Find the equation of a circle referred to a tangent, and a chord through the point of touch.

We have $a_1=b_1=b_2=0,\ a_2=\pm 2r\sin\omega$;

hence, $x^2+y^2+2xy\cos\omega\pm 2rx\sin\omega=0.$

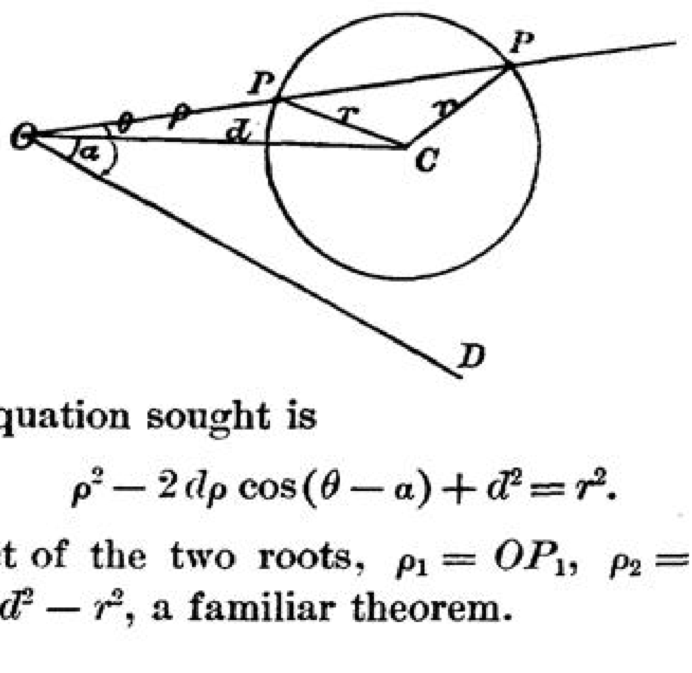
If the chord be a diameter, $\omega=90°$; hence the important form

$$x^2+y^2\pm 2rx=0.$$

Verify geometrically, and explain the double sign.

Polar Equation of the Circle.

62. Be d the tract from the pole to the centre, α its inclination to the polar axis OD, r the radius.



Then the equation sought is

$$\rho^2-2d\rho\cos(\theta-\alpha)+d^2=r^2.$$

The product of the two roots, $\rho_1=OP_1$, $\rho_2=OP_2$, is constant, and $=d^2-r^2$, a familiar theorem.

The Circle in Relation to the Right Line.

63. By Art. 16, a line of first degree (a RL.) cuts a line of second degree (as a circle) in two, and only two, points.

For, in the equation of second degree, put for y (say) its value $sx + b$ (say) taken from the equation of first degree; so we get an equation of second degree in x; its two roots are the x's of the two points common to the two lines.

If $\quad u = (mn - glm + fl^2) : (l^2 + m^2)$,

and $\quad v = \sqrt{\{(mn - glm + fl^2)^2 + (2gln - l^2c - n^2)(l^2 + m^2)\}}$
$: (l^2 + m^2)$

and (x_{1_2}, y_{1_2}) be the common points of the RL. and the circle

$$lx + my + n = 0, \qquad x^2 + y^2 + 2gx + 2fy + c = 0,$$

the student may find (lower index going with lower sign)

$$x_{1_2} = -\frac{n}{l} + \frac{m}{l}(u \pm v), \qquad y_{1_2} = -(u \pm v),$$

but he will not find any pleasure withal. To shun such laborious reckonings and such unmanageable formulas, we have recourse to special forms of the equations.

Thus, the RL. $y = sx + b$ meets the circle $x^2 + y^2 = r^2$ in

$$\left.\begin{aligned} x_{1_2} &= \{-sb \pm \sqrt{r^2(1+s^2) - b^2}\} : \{1 + s^2\} \\ y_{1_2} &= \{b \pm s\sqrt{r^2(1+s^2) - b^2}\} : \{1 + s^2\} \end{aligned}\right\} \qquad \text{(A)}$$

The RL. $x \cos\alpha + y \sin\alpha = p$ meets the circle $x^2 + y^2 = r^2$ in

$$\left.\begin{aligned} x_{1_2} &= p \cos\alpha \pm \sin\alpha\sqrt{r^2 - p^2} \\ y_{1_2} &= p \sin\alpha \mp \cos\alpha\sqrt{r^2 - p^2} \end{aligned}\right\} \qquad \text{(B)}$$

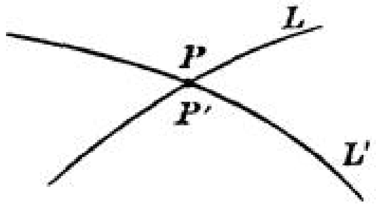
These pairs are *real* and *different*, or *equal*, or *imaginary*; i.e., the common points are *real* and *separate*, or *consecutive*, or *imaginary*, according as

in (A), $\quad r^2(1 + s^2) - b^2$ is > 0, or $= 0$, or < 0;

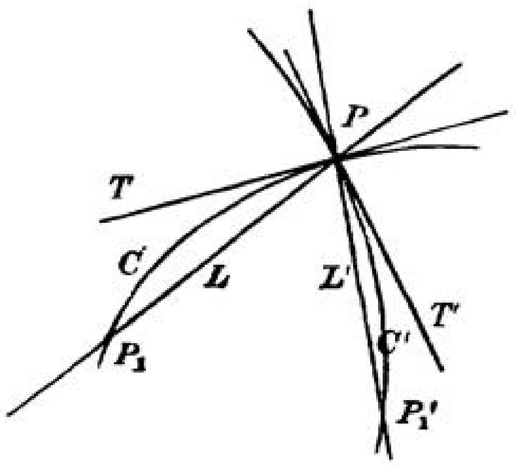
in (B), $\quad r^2 - p^2$ is > 0, or $= 0$, or < 0.

64. *Real* and *separate* points present no difficulty; *imaginary* points have no existence in our plane, and are to be treated further on; but *consecutive* points it is essential to his progress that the student understand clearly now and here.

Coincident points fall together, have exactly the *same position*, and are *distinct* only in *thought*. Thus, we may think of



the intersection of two lines L, L' as made up of two points fallen together, and we may call it P or P' according as we think it belonging to L or to L'. *Consecutive* points *become coincident* and in a particular way: *by nearing each other on the same definite path* (curve). Thus, if C be any curve, P and P_1 any two points on it, we may think P as fixed and P_1 as taken at will nearer and nearer P; or, what comes to the same, we may think P_1 as moving nearer and nearer to P along C. Be L a RL. through P and P_1. As P_1 nears P, L turns about P, and the position of L is fixed completely by P_1. As P_1 falls on P, L turns into some position T. Now P and P_1, thought



simply as *coincident*, cannot fix a RL.: for they form but *one* point, and through this *one* point a RL. may be drawn in *any* direction. But P and P_1, thought as consecutive points (of C say), do fix the position of L; for, as consecutive points of C. they *become* coincident by nearing each other along C. At every stage of this *becoming-coincident*, L turns into a definite position, as P_1 nears P; and at the end of it, the *being*-coincident, as P_1 falls on P, it is left turned into a definite position T. It is specially to note that the *being*-coincident of P and P_1 has no power to fix the position of L; it is only the particular way of their *becoming* coincident that fixes it. As P_1' becomes coincident with P (or P') in some other way: by nearing it along some other curve C', the RL. L' turns into some other position T'.

Looking at the algebraic side of the matter, we find the likeness perfect. The two pairs of common roots (x_1, y_1), (x_2, y_2) of the two Eqs. $y = sx + b$, $x^2 + y^2 = r^2$, are equal if $r^2 = b^2 : (1 + s^2)$; but not simply *are* they equal; they *become* equal in a particular way: not by b and s passing at random through any one of an infinite number of series of pairs of values up to that pair which makes $r^2 = b^2 : (1 + s^2)$, but by their passing through that particular series each one of whose pairs satisfies the two conditions, $y_2 = sx_2 + b$ and $x_2^2 + y_2^2 = r^2$. The whole series of pairs of values of s and b being fixed, the last pair, which makes $r^2 = b^2 : (1 + s^2)$, is fixed; i.e., the last position of the RL. $y = sx + b$ through the consecutive points (x_1, y_1), (x_2, y_2) of $x^2 + y^2 = r^2$, is fixed. Again, we see it is not the fact of *being* coincident, but the way of *becoming* coincident, which is significant. We further see that the concepts, coincident points and consecutive points, are not in themselves complete; we think, though we do not always say: two coincident points of two curves; two consecutive points of one curve.

65. In the light of the above, we may now define a *tangent* to a curve as a *RL. through two consecutive points of the curve.* Where the points fall together is called *point of touch*, *contact*, *tangency*. If for "RL." we put "curve," the definition still holds. We also see that the algebraic condition that a RL. and a curve (or two curves) be tangent is, that two pairs of common roots of their Eqs. be equal.

Hence, if $y = sx + b$ touch $x^2 + y^2 = r^2$, $b^2 = r^2(1 + s^2)$; or,

The RL. $\boldsymbol{y = sx \pm b\sqrt{1 + s^2}}$ *touches the circle* $\boldsymbol{x^2 + y^2 = r^2}$ for all real values of s.

This so-called **magic** equation of the tangent determines it by its direction (s), not by its point of touch, and is useful in problems not involving this point.

So, too, if $x \cos a + y \sin a = p$ touch $x^2 + y^2 = r^2$, $r^2 = p^2$. Hence, the Cds. of the point of touch are

$$x_1 = r \cos a, \quad y_1 = r \sin a.$$

Substituting for $\cos\alpha$, $\sin\alpha$, p in $x\cos\alpha + y\sin\alpha = p$, we get

The RL. $\boldsymbol{xx_1 + yy_1 = r^2}$ *touches the circle* $\boldsymbol{x^2 + y^2 = r^2}$ *at the point* (x_1, y_1) *of the circle.*

This equation determines the tangent by its contact-point (x_1, y_1), and is useful in problems involving that point.

66. The doctrine of **Chords** is so of a piece for all curves of second degree, that it is deemed best to state it here at once in full generality.

We know how to find the intersections of a given RL. with a given curve; the converse would be *to find the RL. through given intersections with the curve.* The general method is this:

Combine the general Eq. of a RL. through two points with two Eqs. which say the given points lie on the given curve; the result will be the Eq. sought.

This tedious general method we may replace by special methods in special cases. Such is the following method of Burnside for the curve of second degree:

Form an Eq. whose terms of degree higher than the first shall cancel (hence, it will picture a RL.), *and which shall be satisfied by the Cds. of the points* (x_1, y_1), (x_2, y_2), *only when these points lie on the curve.*

Such an Eq. formed after this prescription is

$$kx^2 + 2hxy + jy^2 + 2gx + 2fy + c$$
$$= k(x - x_1)(x - x_2) + 2h(x - x_1)(y - y_2) + j(y - y_1)(y - y_2).$$

This, therefore, is the Eq. of a secant through (x_1, y_1), (x_2, y_2) of the curve of second degree. The condition that this secant become a tangent is, that (x_1, y_1), (x_2, y_2) fall together.

$$\therefore kx^2 + 2hxy + jy^2 + 2gx + 2fy + c$$
$$= k(x - x_1)^2 + 2h(x - x_1)(y - y_1) + j(y - y_1)^2,$$

or, after expanding, cancelling, transposing, remembering that

$$kx_1^2 + 2hx_1y_1 + jy_1^2 + 2gx_1 + 2fy_1 + c = 0,$$

finally,

$$kx_1x + h(x_1y + y_1x) + jy_1y + g(x_1 + x) + f(y_1 + y) + c = 0 \quad (\text{D}')$$

is the Eq. of a RL. tangent at (x_1, y_1) *to*

$$kxx + h(xy + yx) + jyy + g(x + x) + f(y + y) + c = 0. \quad (\text{D})$$

The Eq. of second degreé being written thus, the Cds. appear in pairs, and we get the Eq. of the tangent by *substituting for the first current Cd. in each pair the corresponding Cd. of the point of touch.*

These general Eqs. of secant and tangent include all special cases, and are here deduced *once for all.*

EXERCISE.

Form by these methods, then simplify, the Eqs. of chords and tangents of the curves

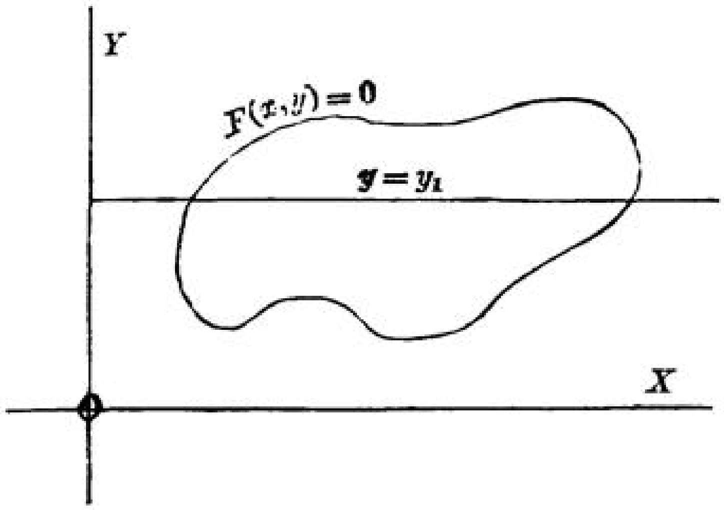
$$xy = c^2, \quad y^2 = 4px, \quad x^2 + y^2 = r^2; \quad (x-a)^2 + (y-b)^2 = r^2.$$

67. *From any point* (x', y') *may be drawn two, and only two, tangents to a curve of second degree.*

For, write $F(x, y; x, y) = 0$ for Eq. (D) of Art. 66; then the Eq. (D$'$) of the tangent is $F(x_1, y_1; x, y) = 0$; and clearly $F(x_1, y_1; x, y) = F(x, y; x_1, y_1)$. If the tangent at (x_1, y_1) go through (x', y'), then $F(x_1, y_1; x', y') = 0$; and, since (x_1, y_1) is on the curve, $F(x_1, y_1; x_1, y_1) = 0$. In the first of these Eqs. x_1, y_1, x', y' all appear linearly; solving as to y_1 (say), we get $y_1 =$ an expression linear in x_1, x'. Substituting this in $F(x_1, y_1; x_1, y_1) = 0$, which is of second degree in x_1 and y_1, we get an expression of second degree in x_1. Solving this, we get two values of x_1, to each of which, since $F(x_1, y_1; x', y') = 0$ is linear in x_1, y_1, corresponds one, and only one, value of y_1; hence, there are two, and only two, pairs of values (x_1, y_1); i.e., two, and only two, points of tangency.

Of course these pairs may be real and unequal, or equal, or imaginary; accordingly, the tangents will be real and separate, or coincident, or imaginary. The second case, of coincidence, arises when the point (x', y') is *on* the curve; for, since the Eq. of a tangent through a point of the curve has been found universally (see Art. 66), there is but *one* such tangent, which may, however, be thought as two fallen together. To tell when they are real, when imaginary, we may reason thus:

68. *Any curve,* $F(x,y)=0$, *bounds off all points of the plane for which* $F(x,y)<0$ *from all points for which* $F(x,y)>0$.



For, assign y any value, say y_1; then, as x varies, $F(x, y)$ will become 0 only where the RL. $y=y_1$ cuts the curve $F(x,y)=0$; but, as, and only as, x passes through a root x_1 of $F(x,y_1)=0$, $F(x,y_1)$ changes sign. We may call the sides of the curve plus resp. minus.

In the Eqs. of the curve and the tangent, (D) and (D′), let us replace 1 by v resp. v_1, so as to make them homogeneous in x, y, x_1, y_1, v, v_1. They then take the forms

$$kxx+h(xy+yx)+jyy+g(xv+vx)$$
$$+f(yv+vy)+cvv=0, \qquad \text{(E)}$$

$$kx_1x+h(x_1y+y_1x)+jy_1y+g(x_1v+v_1x)$$
$$+f(y_1v+v_1y)+cv_1v=0. \qquad \text{(E}'\text{)}$$

If, now, we proceed as sketched in Art. 67 to find x_1, we shall get a result of the form

$$Ax_1 + Bv_1 = \pm\sqrt{C},$$

where A, B, C are functions of x', y', v', k, h, j, g, f, c.

Since $$y_1 = -\frac{(kx' + hy' + gv')_1 x_1 + (gx' + fy' + cv')_3 v_1}{(hx' + jy' + fv')_2},$$

we readily see that C will be homogeneous of tenth degree in all the arguments: of fourth degree in x', y', v', and of sixth degree in k, h, j, g, f, c. [Consider that in replacing y_1 the parentheses $(\ )_1$, $(\ )_2$, $(\ )_3$ will be squared, which squares will be again squared in completing the square (in x_1, y_1), as will the coefficients k, h, etc.] Now, if (x', y') be on $F(x, y\,;\,x, y) = 0$, the two values of x_1 are equal; but then $C = 0$; hence, $F(x', y'\,;\,x', y') = 0$ makes $C = 0$; i.e., $F(x', y'\,;\,x', y')$ is a factor of C. Now, since x' enters $F(x', y'\,;\,x', y')$ in second degree, but enters C only in fourth degree, it follows that $F(x', y'\,;\,x', y')$ cannot appear in C in higher than second degree. If it appear in second degree, we may extract the second root, and write

$$\sqrt{C} = F(x', y'\,;\,x', y')\sqrt{R}.$$

Here R cannot contain x', y', or v', since each enters F in second degree, and each entered C in only fourth degree; hence R is a function of k, h, j, g, f, c only, and that of second degree. Hence, whether $\sqrt{C}$ be real or imaginary will depend only on the sign of R, that is, only on k, h, j, f, g, c, not at all on x', y'; that is, not at all on the position of the point (x', y') from which the tangents are drawn. Hence, tangents from all points will be either all real or all imaginary. This is so for any curve of second degree, hence for the special curve, circle. But now we know that some tangents to the circle (from points outside the circle) are real, while some (from points inside) are imaginary. Hence, $F(x', y'\,;\,x', y')$ cannot enter C in second degree, but only in first degree. $F(x', y'\,;\,x', y')$ changes sign at every point of the curve $F(x, y\,;\,x, y) = 0$, and no other function of x and y does. Hence, C changes sign along the same curve, is plus for all points (x', y') on one side, minus for all points (x', y') on the other; hence, tangents from all points on one side of the curve are real, from all points on the other side are imaginary. The side on which the tangents are real, we may call the outside; the other, the inside.

69. The Eq. $F(x_1, y_1\,;\, x, y) = 0$, of the tangent to $F(x, y\,;\, x, y) = 0$ is symmetric as to x and x_1, as to y and y_1, a fact of highest import to the whole theory of curves of second degree. This import we shall now in part develop.

Thus far, the point (x_1, y_1) has been taken on the curve; the query is natural, What does $F(x_1, y_1\,;\, x, y) = 0$ picture when (x_1, y_1) is not on the curve? To answer it, suppose tangents drawn from (x_1, y_1) touching the curve at (x_2, y_2), (x_3, y_3). The Eq. of one is $F(x_2, y_2\,;\, x, y) = 0$.

Since it goes through (x_1, y_1), $F(x_2, y_2\,;\, x_1, y_1) = 0$. But this Eq. also says that $F(x, y\,;\, x_1, y_1) = 0$ goes through (x_2, y_2); by like reasoning, we show that it goes through (x_3, y_3); hence,

$$\boldsymbol{F(x, y\,;\, x_1, y_1) = 0} \quad \text{or} \quad \boldsymbol{F(x_1, y_1\,;\, x, y) = 0}$$

is the RL. through the tangent points of tangents from (x_1, y_1) *to* $F(x, y\,;\, x, y) = 0$.

Such a RL. is named **polar** of the **pole** (x_1, y_1) as to $F(x, y\,;\, x, y) = 0$.

Since the equation of condition $F(x_1, y_1\,;\, x_2, y_2) = 0$ may be read either

(x_1, y_1) *is on the polar of* (x_2, y_2), i.e., on $F(x, y\,;\, y_2, y_2) = 0$,

or (x_2, y_2) *is on the polar of* (x_1, y_1), i.e., on $F(x_1, y_1\,;\, x, y) = 0$,

if one pole be on the polar of a second, the second is on the polar of the first: or, if one polar pass through the pole of a second, the second passes through the pole of the first.

Two poles, each on the polar of the other, or two polars, each through the pole of the other, are called **conjugate**.

Hence, if a point be on each of a system of polars, i.e., be their intersection, the poles of each polar will be on the polar of the point; or, as a RL. turns around a point, its pole glides along the polar of the point; or, as a point glides along a RL., its polar turns about the pole of the RL.

If we convert the definition of the polar of a point, we shall get a definition of the pole of a RL.: *as the intersection of the tangents to* $F(x, y; x, y) = 0$ *through the intersection of the RL. and the curve.* If, now, this RL. turn about a point, its pole will glide along the polar of the point; hence, *the polar of a point is the locus of the intersection of the pair of tangents to the curve through the intersection with the curve of a RL. through the point.*

70. These two definitions of *polar* are equivalent. The first yields a geometric construction only when the pole is *outside* the curve, for only then are the tangent points real. If the point be *inside* the curve, we may still draw through it two chords of the curve, and draw the two pairs of tangents through their ends; the RL. through the two intersections of the tangent pairs will then be the polar sought. When the pole falls on the curve, the Eq. shows the polar becomes a tangent through its pole as point of tangency. Hence, the *tangent is* to be thought as *a polar through its own pole.*

It is carefully to note that the terms *pole* and *polar* are meaningless without reference, express or implied, to a curve of second degree, which we may call the **referee**.

The notions of pole and polar are still deeper inwrapped in the notion of curve of second degree, as what follows may show.

71. Be $\mu_1 : \mu_2$ the ratio in which $F(x, y; x, y) = 0$ cuts the tract $\overline{(x_1, y_1)(x_2, y_2)}$; then the Cds.

$$x = \frac{\mu_1 x_2 + \mu_2 x_1}{\mu_1 + \mu_2}, \qquad y = \frac{\mu_1 y_2 + \mu_2 y_1}{\mu_1 + \mu_2}$$

of the section-point must satisfy $F(x, y; x, y) = 0$. Substituting herein and arranging terms, we get

$$\mu_1^2 F(x_2, y_2; x_2, y_2) + 2\mu_1\mu_2 F(x_1, y_1; x_2, y_2) + \mu_2^2 F(x_1, y_1; x_1, y_1) = 0. \tag{G}$$

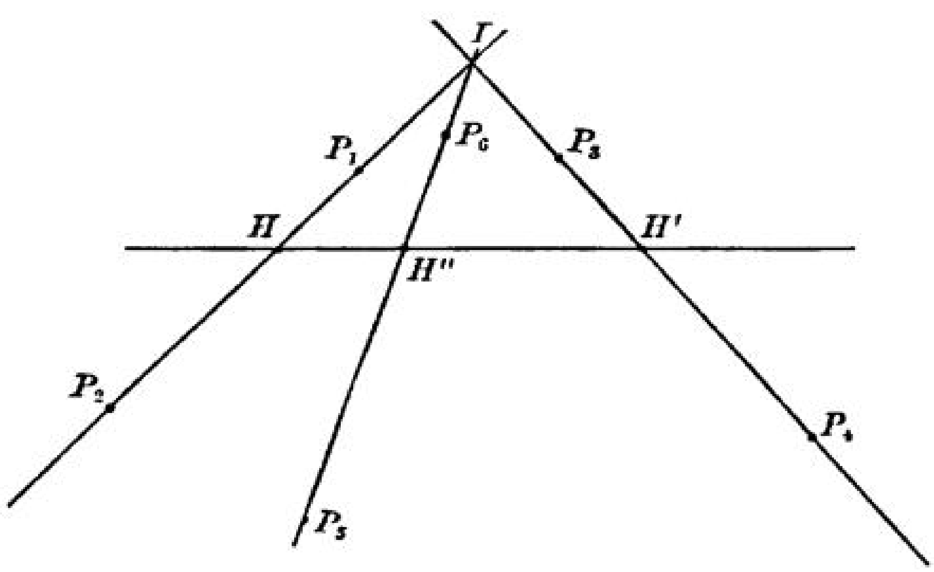
[N.B. Reason as in Art. 51; the result must be homogeneous in μ_1 and μ_2, and symmetric as to indices 1, 2; the coefficient of μ_1^2 resp. μ_2^2 we get by supposing μ_2 resp. μ_1 to be 0; the coefficient of $\mu_1\mu_2$ must be double and symmetric as to the indices 1, 2.]

This quadratic yields two values of the ratio $\mu_1 : \mu_2$, say r' and r''. If (x_1, y_1) and (x_2, y_2) be *conjugate* (each on the other's polar), then $F(x_1, y_1; x_2, y_2) = 0$, and then $r' = -r''$; i.e., the tract $(x_1, y_1)(x_2, y_2)$, from pole to polar, is cut by $F(x, y; x, y) = 0$ innerly and outerly in like ratio; i.e., is cut harmonically. Hence, *any tract from a pole to its polar is cut harmonically by the referee.*

Hence, once again, we may define *polar* thus:

The locus of the harmonic conjugate to a fixed point, the other pair being section-points with the referee of chords through the point, is called the **polar** *of the fixed point* (as to the referee).

Thus is justified the use of *conjugate* in Art. 69.



If, now, there be given five points of the referee, we may construct it with the ruler only, thus: Through P_1 and P_2, P_3

and P_4 draw secants meeting in I; on them find fourth harmonics * H, H' conjugate to I; HH' is polar of I. Draw IP_5 cutting HH' at H''. Then is P_6, the fourth harmonic to P_5, the other pair being I, H'', a point of the Referee. As we may choose four out of five in five ways, and join each four by twos in three ways, we may thus construct fifteen sixth-points. Recombining these twenty, we may go on to construct any number of points of the *Referee.*

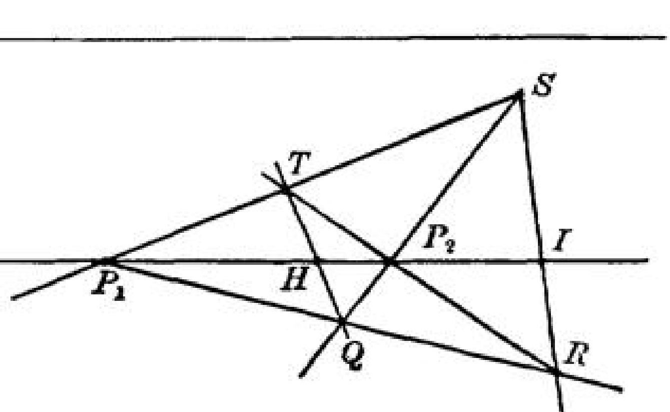
72. In Eq. (G) drop the subscript $_2$; then the roots r', r'' are the ratios in which $F(x, y; x, y) = 0$ cuts the tract from (x_1, y_1) to (x, y). When these ratios are equal, the tract is cut by the curve in two consecutive points; i.e., the RL. through (x_1, y_1) and (x, y) touches the curve; but when the roots are equal,

$$F(x_1, y_1; x_1, y_1) \cdot F(x, y; x, y) = \{F(x_1, y_1; x, y)\}^2.$$

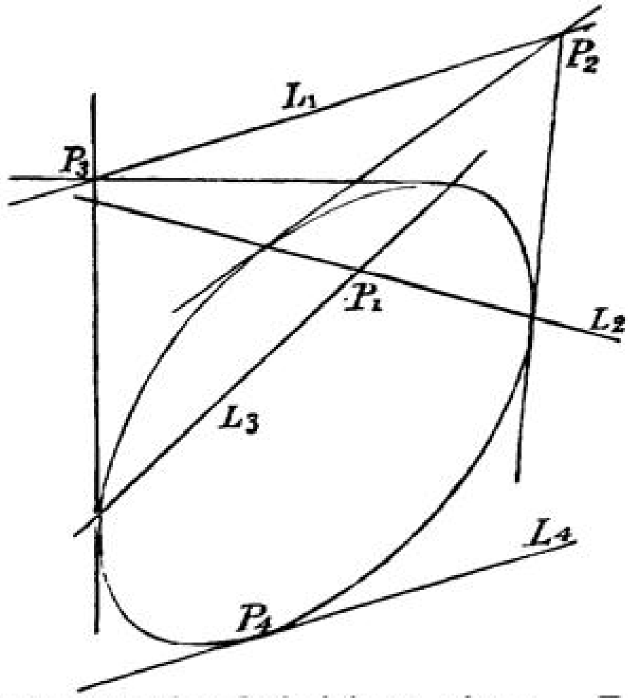
In this Eq. (x, y) is any point on a RL. through (x_1, y_1) tangent to $F(x, y; x, y) = 0$; hence, *this Eq.*, being of second degree in (x, y), *pictures the pair of tangents through* (x_1, y_1) *to* $F(x, y; x, y) = 0$.

The right member being a square, is always plus; hence, the factors on the left are like-signed; hence all points (x, y) of a tangent lie on the same side as any one point (x_1, y_1); i.e., *the tangent does not cut the curve;* i.e., the curve is throughout convex or concave on the same side; not like this figure.

* This we may do by thinking $P_1 P_2$ a diagonal of a four-side cut by another diagonal at I. Draw SP_1, SP_2, SI; draw P_1QR at will; draw RP_2T; draw QT; it cuts P_1P_2 at H, by Art. 49.



Thus far all geometric representation has been purposely avoided, to show more clearly how the notions and properties of pole and polar all lie enfolded in the algebraic fact that in the Eq. of the tangent to the curve of second degree current Cds. and Cds. of the point of touch appear symmetrically.



The figure illustrates the definitions given. Poles and polars are marked by the letters P and L, with corresponding indices. It is not necessary to know aught of the curve except that it is of second degree.

We may now return to the special properties of the circle.

73. By Art. 63, the section-points of $x \cos a + y \sin a = p$ and $x^2 + y^2 = r^2$ are

$$x_{1_2} = p \cos a \pm \sin a \sqrt{r^2 - p^2},$$

$$y_{1_2} = p \sin a \mp \cos a \sqrt{r^2 - p^2}.$$

The half-sums of these pairs are the Cds. of the mid-point of the intercepted chord: $x_m = p \cos a$, $y_m = p \sin a$.

By eliminating p we get a relation holding between the Cds. of the mid-points of all chords having the same a, i.e., all $\parallel$ chords. Hence,

$$y_m : x_m = \tan a \quad \text{or} \quad y = \tan a \cdot x$$

is the locus of the mid-points of a system of ∥ chords. Such a locus is called a **diameter**, and in this case is clearly a *RL. through the centre ⊥ to the chords.*

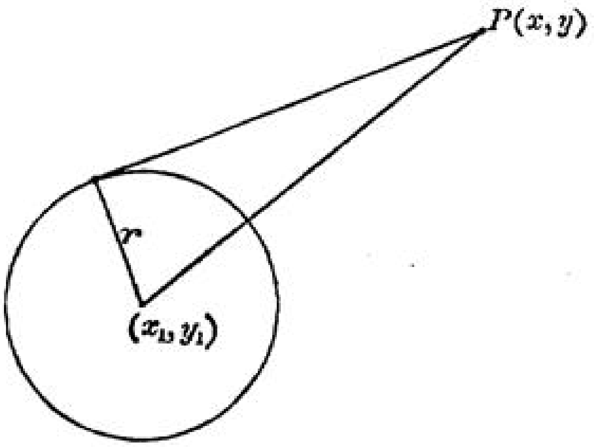
A RL. through any point (x_1, y_1) of a curve ⊥ to the tangent at that point is named **Normal** to the curve at that point.

The tangent to $x^2+y^2=r^2$ at (x_1, y_1) is $xx_1+yy_1=r^2$; hence,
$$(x-x_1)y_1-(y-y_1)x_1=0, \quad \text{or} \quad x:y=x_1:y_1,$$
is normal to the circle $x^2+y^2=r^2$ at (x_1, y_1).

The absolute in this Eq. is 0; hence,

Normals to a circle pass through (envelop) *a point*, the centre. Also, all normals are diameters of a circle.

74. In the expression
$$\overline{x-x_1}^2+\overline{y-y_1}^2-r^2, \quad \text{or} \quad x^2+y^2+2gx+2fy+c,$$
either of which equivalents equated to 0 is the N.F. of the rectang. Eq. of the circle whose centre is (x_1, y_1), and radius r, $\overline{x-x_1}^2+\overline{y-y_1}^2$ is the squared distance of any point (x, y) from the centre; and since the radius is ⊥ to the tangent at its end (Art. 73), the difference $\overline{x-x_1}^2+\overline{y-y_1}^2-r^2$ is the squared tangent-length from (x, y) to the circle $\overline{x-x_1}^2+\overline{y-y_1}^2-r^2=0$.



So, too, is $x^2+y^2+2gx+2fy+c$ the squared tangent-length from (x, y) to the circle $x^2+y^2+2gx+2fy+c=0$.

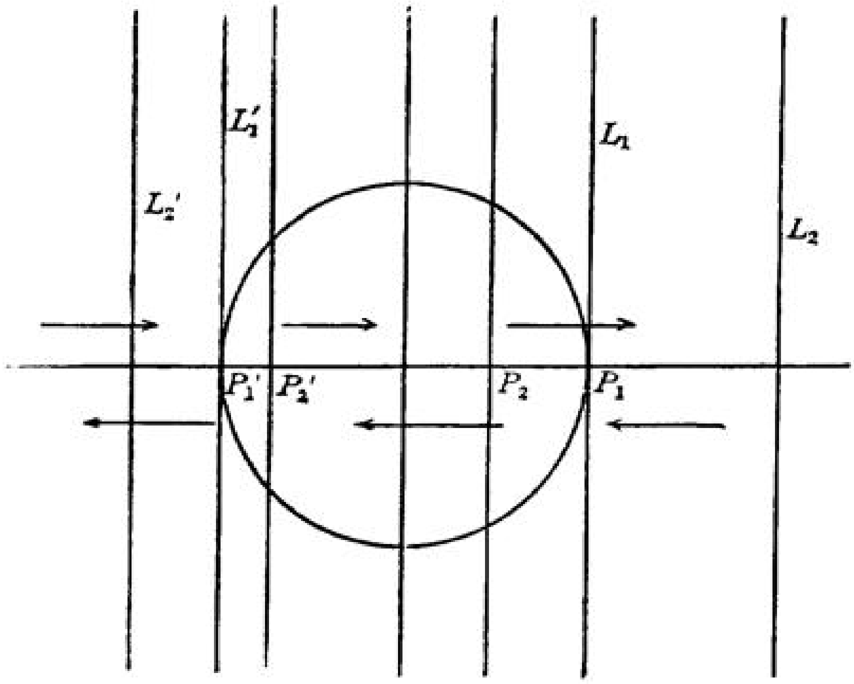
This *squared tangent-length* is called the **power** *of the point* (x, y) as to the circle. To find it, replace the current Cds. in the N.F. of the Eq. of the circle by the Cds. of the point.

Like reasoning and results hold for oblique axes. The power of the origin (0, 0) is the absolute in the N.F. of the Eq.

75. The centre being origin, the Eq. of the circle is $x^2+y^2=r^2$, however the axes be turned. Turn them till the X-axis passes through any assumed pole. Then is $y_1 = 0$, and the polar is $xx_1 = r^2$ or $x = \frac{r^2}{x_1}$.

But this is a RL. ⊥ to X-axis, distant x from origin.

Hence: (1) The circle being referee, *the polar is ⊥ to the radius through the pole; (2) the radius is the geometric mean of the distances of pole and polar from the centre.*



We may now trace the movement of pole and polar thus: When the pole P is at P_1, the polar L is the tangent L_1; when P falls on P_2, L falls on L_2; as P nears the centre, L retires to ∞; as P passes through the centre, L passes through ∞; as P nears P_1', L nears L_1'; as P retires to ∞, L nears the centre; as P passes through ∞, L passes through the centre; as P nears P_1, L nears L_1. So long as P stays on a diameter, L stays ∥ in all positions; i.e., L turns around its point ∞, also P and L move counter. As P glides along a tangent, L turns about the point of tangency; as P glides along any RL., L turns about the pole of that RL.; as P glides around any circle concentric with the referee, radius a, L turns around a second concentric circle, radius $\frac{r^2}{a}$; if P glide around this second circle, L will turn around the first. All these are special cases of the general proposition: *as the pole glides along* (traces) *any curve of nth degree, the polar turns around* (envelops, enwraps) *a curve of nth class; and conversely.* Such pairs of curves are called *reciprocal.*

N.B. A curve cut by a RL. in n points is of nth degree; a curve touched by n RLs. through a point is of nth class. Curves of second degree and curves of second class are the same; but in general curves do not rank alike in degree and class. See Art. 160.

EXERCISES.

1. Find tangents to $x^2 + y^2 - 6x - 14y - 3 = 0$ at the points whose x is 9.

From the Eq. of the circle we find the corresponding value of y: 12, 2. The Eq. of the circle in the form $F(x, y; x, y) = 0$ is $xx + yy - 3(x + x) - 7(y + y) - 3 = 0$. Hence, the tangents are

$$9x + 12y - 3(9 + x) - 7(12 + y) - 3 = 0,$$
$$9x + 2y - 3(9 + x) - 7(2 + y) - 3 = 0;$$

or, reduced, $\quad 6x + 5y = 114, \quad 6x - 5y = 44.$

2. Similarly, find the tangents thus defined:

$$x^2 + y^2 - 4x + 22y + 25 = 0, \quad x_1 = 3;$$
$$(x - 5)^2 + (y + 8)^2 = 113, \quad x_1 = 13.$$

3. Find the tangents to $x^2 + y^2 + 10x - 6y - 2 = 0$ ∥ to $y = 2x - 7$.

The Eq. of the circle may be written $(x + 5)^2 + (y - 3)^2 = 36$; or, if $x' = x + 5$, $y' = y - 3$, $x'^2 + y'^2 = 36$. The Eq. of the RL. becomes $y' = 2x' - 20$. A RL. ∥ must be $y' = 2x' + b$. This is tangent to $x'^2 + y'^2 = 36$ when, and only when, $36(1 + 2^2) = b^2$; i.e., when $b = \pm 6\sqrt{5}$. Therefore the tangents are $y' = 2x' \pm 6\sqrt{5}$; or, $y = 2x + 13 \pm 6\sqrt{5}$.

4. Draw tangents to $x^2 + y^2 = 58$ inclined 60° to $4x - 3y = 12$.

5. Through $(x_1 = 9, y_1 > 0)$ on $x^2 + y^2 - 12x + 2y + 3 = 0$ draw RLs. inclined 45° to the circle.

Hint. The RLs. sought halve the angles between tangent and normal.

6. Find the angle between two tangents to a circle.

7. Find the power of $(-11, -9)$ as to $(x - 3)^2 + (y - 7)^2 = 25$; of $(4, 1)$ as to $4x^2 + 4y^2 - 3x - y - 7 = 0$.

8. Find the circle tangent to $y = 3x - 5$, centre at origin.

9. Find the tangents from $(16, 11)$ to $x^2 + y^2 = 169$, and where they touch.

10. Find and draw the polars of: $(11, 17)$ as to $(x-3)^2+(y+5)^2=81$; $(8, -5)$ as to $x^2+y^2+14x+6y+22=0$; $(-2, -7)$ as to $x^2+y^2-18x+2y+57=0$.

11. Find the polar of (x_1, y_1) as to the point (-circle) (a, b).

The Eq. of the point regarded as circle of vanishing radius is

$$(x-a)^2+(y-b)^2=0;$$

$\therefore$ the polar of (x_1, y_1) is $(x_1-a)(x-a)+(y_1-b)(y-b)=0$.

This RL. goes through (a, b) $\perp$ to the junction-line of (x_1, y_1) and (a, b). Show that the polar of $(11, 3)$ as to $(4, -2)$ is $7x+5y-18=0$.

12. Find the polar of (x_1, y_1) as to a RL.

Regard the RL. as circle of infinite radius, and write its Eq.

$$(x^2+y^2)(1+k)+2(A_1+kA_2)x+2(B_1+kB_2)y+C_1+kC_2=0.$$

For $k=-1$ this circle passes over into a RL. The polar of (x_1, y_1) is,
for $k=-1$, $(A_1-A_2)(x_1+x)+(B_1-B_2)(y_1+y)+C_1-C_2=0$,
the RL. is $2(A_1-A_2)x+2(B_1-B_2)y+C_1-C_2=0$.

Hence, the polar is $\parallel$ to the RL., midway between it and the pole.

Show that the polar of $(8, 20)$ as to $5x-3y+7=0$ is $5x-3y-6=0$.

13. What is the pole: of $y=mx+b$ as to $x^2+y^2=r^2$?
of $Ax+By+C=0$ as to $(x-a)^2+(y-b)^2=r^2$?

14. Find the pole (x_2, y_2) conjugate to (x_1, y_1) as to $x^2+y^2=r^2$ and on the junction-line of (x_1, y_1) and the centre of the circle.

Since (x_2, y_2), as conjugate to (x_1, y_1), lies on the polar of (x_1, y_1), $x_1x_2+y_1y_2=r^2$; since it lies on $x:y=x_1:y_1$, $x_2:y_2=x_1:y_1$.

Hence, $x_2=r^2x_1:(x_1^2+y_1^2)$, $y_2=r^2y_1:(x_1^2+y_1^2)$.

Hence, $x_1=r^2x_2:(x_2^2+y_2^2)$, $y_1=r^2y_2:(x_2^2+y_2^2)$.

Hence, (x_1, y_1) and (x_2, y_2) are called **related** as to $x^2+y^2=r^2$.

15. Show that each of two related points is the pole of a RL. through the other $\parallel$ to the polar of the other.

Systems of Circles.

76. Be $C_1=0$, $C_2=0$ Eqs. of two circles in normal form.

Then is $C_1-\lambda C_2=0$ *the Eq. of a circle* (by Art. 58) *through the common points of* $C_1=0$ *and* $C_2=0$ (by Art. 30).

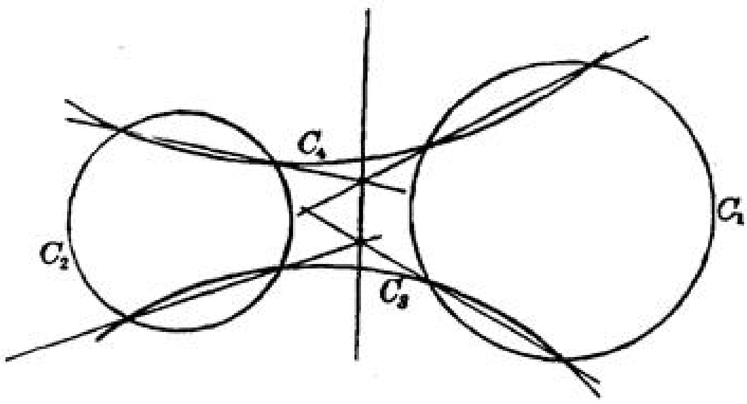
Or, $C_1 - \lambda C_2 = 0$ *is the Eq. of a system or family of circles through two fixed points: the section-points of* $C_1 = 0$ *and* $C_2 = 0$; λ is the *parameter* of the system, and ranges from $-\infty$ to $+\infty$.

Since C_1 and C_2 are the powers of any point as to $C_1 = 0$ respectively $C_2 = 0$, and since λ is clearly the ratio of these powers, we see that

The ratio of the powers of any point on any circle of the system as to any two circles of the system is constant. The ratio is of course different for different circles or as to different pairs of circles.

For $\lambda = 1$ the terms of second degree vanish, the circle passes over into a RL., i.e., a circle of infinite radius. This RL. is always real, though the section-points be imaginary. Clearly *the powers of its every point as to the two circles are equal*, since its Eq. is $C_1 - C_2 = 0$. Hence it might be named *Equipotential Line* or simply *Power-Line* of the circles (called *Radical Axis* by Gaultier, 1813).

77. The power-lines of three circles taken in sets of two are $C_1 - C_2 = 0$, $C_2 - C_3 = 0$, $C_3 - C_1 = 0$; added, these Eqs. vanish identically; i.e., *the three power-lines meet in a point* — the **power-centre** of the three circles.



Hence, to construct the power-line of two circles, C_1 and C_2, draw C_3 and C_4 cutting C_1 and C_2. The power-centres of C_1, C_2, C_3 and C_1, C_2, C_4 fix two points on the power-line of C_1 and C_2.

78. Two points determine a RL. as common power-line of a system of circles: $C_1 - \lambda C_2 = 0$, through the points. *The power-lines of each circle of such a system and a fixed circle C pass through a point.* For the power-line of C and any circle C' of the system cuts the given power-line of the system say at I; which then is the power-centre of C, C' and any second circle C'' of the system; hence the power-line of C and C'' also passes through I.

EXERCISES.

1. Find in co-ordinates the power-centre of $(x-7)^2+(y-9)^2=36$, $(x+3)^2+(y-2)^2=16$, $(x+4)^2+(y+5)^2=9$, and draw the figure.

2. Show that the power-line of two circles is $\perp$ to the junction-line of their centres (or *centre-line*, as it may be called).

79. The form $C_1 - \lambda C_2 = 0$ is not convenient for studying a system of circles. The power-line and junction-line of centres, being $\perp$, naturally suggest themselves as axes. The latter being taken for X-axis, the term in y falls away; also, for $x=0$ the values of y are equal and unlike-signed for all the circles; hence the parameter λ can enter only the term in x, and we can write the Eq. of the system,

$$x^2+y^2-2\lambda x+\delta^2=0.$$

Here λ is the changing distance of the centre from the origin; δ is the fixed distance to the section-points from the origin; these points are real or imaginary, according as δ^2 is $-$ or $+$.

80. The Eq. of the system of polars of any point (x_1, y_1) as to this system of circles is $x_1x+y_1y-\lambda(x_1+x)+\delta^2=0$; λ appears in first degree only, hence this system of polars pass through a point, the section of $x_1x+y_1y+\delta^2=0$ and $x_1+x=0$.

In general, then, the polar of a point changes with the circle of the system, turning about a point; but if the two RLs. which fix this point, $x_1x+yy_1+\delta^2=0$, $x_1+x=0$, be the *same*

RL., then are all RLs. of the system $x_1 x + y_1 y - \lambda(x_1 + x) + \delta^2 = 0$ the *same* RL.

This is so when, and only when, $y_1 = 0$, $x_1 = \pm\,\delta$, for otherwise the two Eqs. are not the same. Hence *each* of these *two* and *only* these *points* $(\delta, 0)$, $(-\delta, 0)$ has the *same* polar as to all circles of the system, namely, a RL. through the other ⊥ to the line of centres.

These points are *real* or *imaginary* according as δ^2 is + or −, i.e., according as the section-points of the circles are *imaginary* or *real*. Writing the Eq. of the system thus,

$$y^2 + (x - \lambda)^2 = \lambda^2 - \delta^2,$$

we see if δ^2 be +, and so the above critical points real, then the circle is imaginary, for every $\lambda < \delta$. For λ very large the centre retires toward ∞ along the X-axis, the circle flattens toward the Y-axis; as λ nears δ, the centre nears the critical point $(\delta, 0)$, the circle shrinks toward and around that point; and as λ equals δ, the circle *vanishes* in that point. Hence the critical points $(\delta, 0)$, $(-\delta, 0)$ are themselves circles of the system, point-circles, and are hence named by Poncelet *limiting points* of the system.

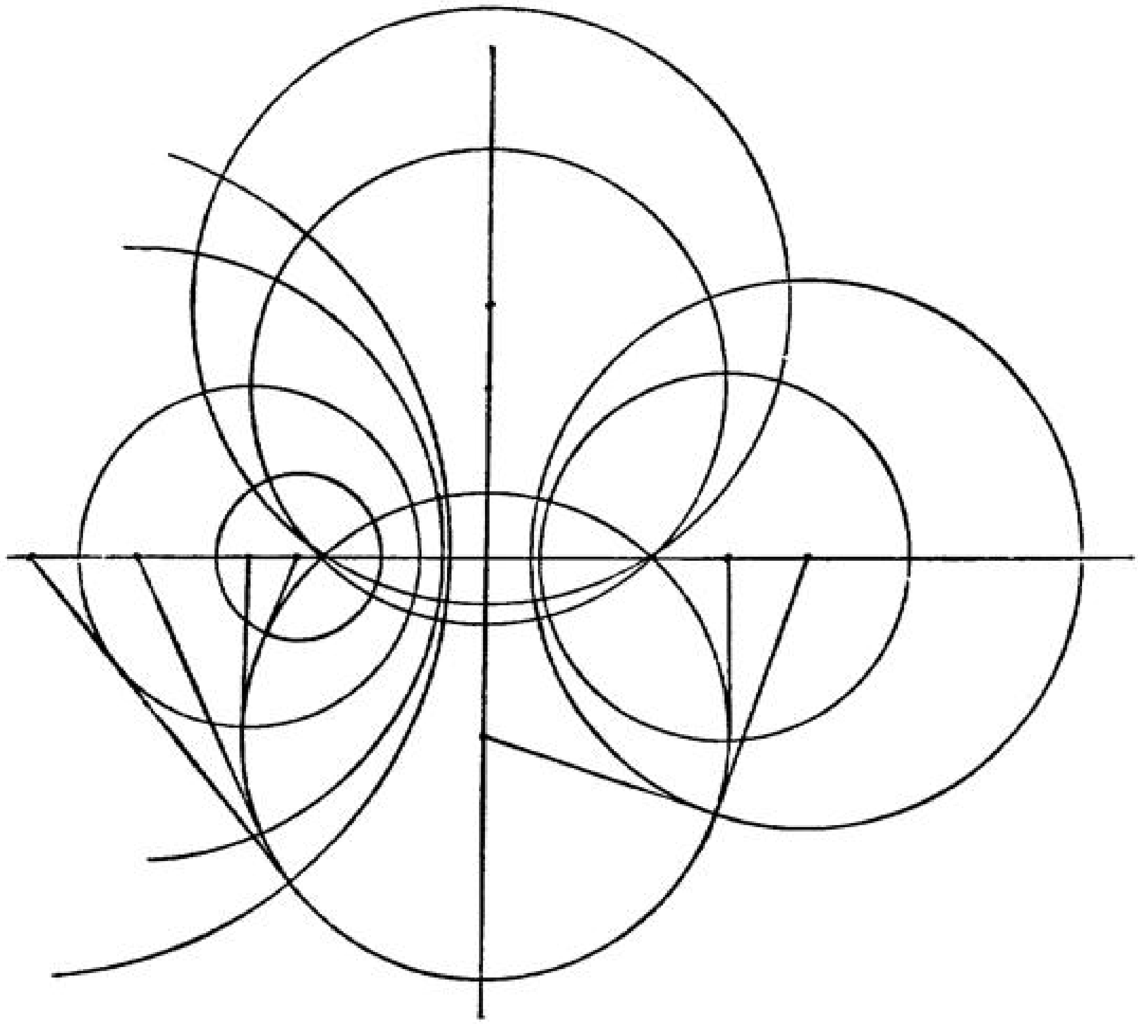
81. The powers of any point of the common power-line as to these circles, i.e., the squared tangent-lengths from any point of the power-line to the circles, are all equal; i.e., the *ends* of all such tangent-tracts, the points of tangency, *lie on a circle* with centre on the common power-line. The radii of this circle, as tangent to the other circles, are ⊥ to the radii of those circles; i.e., *each circle with centre a point of the power-line and squared radius the power of that point cuts orthogonally the whole system of circles.*

As the limiting points are circles of the system, *each orthogonal circle passes through the limiting points.* Hence the Eq. of the system of orthogonal circles is

$$x^2 + y^2 - 2\lambda y - \delta^2 = 0 = x^2 + (y - \lambda)^2 - \lambda^2 - \delta^2.$$

Deduce this directly as the Eq. of the orthogonal system.

Note that these two mutually orthogonal systems are complementary: in the one δ^2 is $+$, in the other $-$; the power-line of one is the centre-line of the other; the section-points of one are the limiting points of the other; of the one the section-points are imaginary, the limiting points real, — of the other, *vice versa*.



To construct this double system. Draw any number of circles through two points. To any one draw any number of tangents. About the points where these cut the RL. of the two fixed points describe circles with the tangent-tracts as radii. Do not fail to carry out this construction.

EXERCISES.

1. When are $C=0$ and $C_1=0$ $\perp$?

The squared distance between their centres must equal the sum of their squared radii; i.e.,

$$(g-g_1)^2+(f-f_1)^2=g^2+f^2-c+g_1^2+f_1^2-c_1,$$

or, $$2gg_1+2ff_1-c-c_1=0. \qquad \text{(A)}$$

Hence, if $C=0$ cut both $C_1=0$ and $C_2=0$ at right angles, we have

$$2g_1g+2f_1f-c-c_1=0 \quad \text{and} \quad 2g_2g+2f_2f-c-c_2=0.$$

From these we may express any two of the three symbols g, f, c through the other linearly; substituting in $C=0$ we get an Eq. of a circle containing one parameter linearly; hence all circles cutting two circles orthogonally form a system through two points, i.e., with common power-line, as already proved.

If $C=0$ cut three circles orthogonally, then we have three Eqs. of the form (A), which with $C=0$ give, through elimination of g, f, c, as the equivalent of $C=0$:

$$\begin{vmatrix} x^2+y^2 & -x & -y & 1 \\ c_1 & g_1 & f_1 & 1 \\ c_2 & g_2 & f_2 & 1 \\ c_3 & g_3 & f_3 & 1 \end{vmatrix}=0.$$

Geometrically this is clearly a circle about the power-centre, its squared radius the power of that centre.

2. The circle orthogonal to $C_1=0$, $C_2=0$, $C_3=0$ is also orthogonal to $\lambda_1C_1+\lambda_2C_2+\lambda_3C_3=0$. Use condition (A).

3. The polar of one end of a diameter of a circle as to any orthogonal circle passes through the other end.

Be $x^2+y^2=r^2$ the given circle; then by (A) an orthogonal circle is $x^2+y^2+2gx+2fy+r^2=0$. As to this the polar of $(r, 0)$ is

$$rx+g(r+x)+r^2=0,$$

which always goes through $(-r, 0)$.

4. Powers of points of one circle as to another vary as their distances from the power-line.

5. To find the angle α_1, under which $C=0$ and $C_1=0$ intersect. α_1 equals the angle between the radii to a section-point. If r, r_1 be the radii, d the distance between the centres, then

$$2rr_1\cos\alpha_1=r^2+r_1^2-d^2.$$

If we hold $C_1=0$ fixed and α_1 constant, then since d^2-r^2 is the squared tangent-length to $C_1=0$ from the centre of $C=0$, we have this relation between the Cds. of the centres of all circles, $C=0$, which cut $C_1=0$, under the angle α_1:

$$r^2-2r_1r\cos\alpha_1=C_1. \qquad \text{(H)}$$

Since x and y enter C_1, this Eq. contains three arbitraries, x, y, r. Imposing the further condition that $C=0$ cut $C_2=0$ under c_2, we get a second Eq.: $r^2-2r_2\cos\alpha_2=C_2$. These two Eqs. do not yet fix the centre (x, y) and the radius r of $C_1=0$; but they determine its family. For it will cut any circle of the system $C_1-\lambda C_2=0$ under a constant angle. We have, namely,

$$r^2-2r\cdot\frac{r_1\cos\alpha_1-\lambda r_2\cos\alpha_2}{1-\lambda}=\frac{C_1-\lambda C_2}{1-\lambda},$$

which declares that the varying circle $C=0$ cuts the circle

$$C'=\frac{C_1-\lambda C_2}{1-\lambda}=0,$$

radius r', under an angle γ such that

$$r'\cos\gamma=(r_1\cos\alpha_1-\lambda r_2\cos\alpha_2):(1-\lambda).$$

We may express r' through the constants of C_1 and C_2 thus:

$$r'^2=\{(1-\lambda)(r_1{}^2-\lambda r_2{}^2)-\lambda d^2\}:(1-\lambda)^2.$$

Hence γ is determined univocally, i.e., is constant.

If we assign γ at will and substitute the value of r', we get a quadratic for determining λ; i.e., there are *two* circles of the system $C_1-\lambda C_2=0$ which the varying circle $C=0$ cuts under any given angle. As a special case, for $\gamma=0$, $\cos\gamma=1$, we see that there are two circles of the system which the varying circle always touches, on which it *rolls*.

6. Through the section-points of $x^2+y^2+4x-14y-68=0$ and $x^2+y^2-6x-22y+30=0$ draw a circle tangent to X-axis.

7. Under what angle do $x^2+y^2=16$ and $(x-5)^2+y^2=9$ intersect?

8. Find the power-centre of

$$x^2+y^2-2x+6y-15=0,$$
$$x^2+y^2+14x+12y+81=0, \qquad \text{and the point } (3, -7).$$

9. Find a circle through $(-5, -4)$ and cutting orthogonally

$$x^2+y^2-4x-6y+9=0, \quad x^2+y^2+6x-4y+4=0.$$

10. Find a circle through $(-4, 3)$, $(-2, -3)$ cutting $x^2+y^2-6x-7=0$ orthogonally.

11. Show that the point-circles cut a diameter of every circle of the system harmonically.

12. What circle of $x^2+y^2-2\lambda x+\delta^2=0$ cuts $(x-a)^2+(y-b)^2=r^2$ orthogonally?

13. It has been shown (Art. 80) that the polars of a point as to a system of circles pass through a point, — the centre of the polar family. These two points are called *poles harmonic* as to the system of circles. Show that the power-line halves the distance between any two harmonic poles, and that the circles cut the junction-line of the poles in an involution of points.

14. Find a circle whose power-lines with two given circles go through their centres.

15. Show that the centres of all such circles lie on a RL. ∥ to the power-line of the two circles, named *secondary power-line.*

16. Show that the circles halving the circumferences of two given circles form a system, and find the power-line.

17. Show that the secondary power-lines of three circles go through a point.

18. Find a circle halving the circumferences of the three circles:
$x^2+y^2-2x-9=0$, $x^2+y^2+3x-9=0$, $x^2+y^2-6y-10x+18=0$.

Centres and Axes of Similitude.

82. Congruent figures are alike in shape and size, differ only in place; they may be thought fitted one on another.

Similar figures are alike in shape, but not in size. Such figures may be supposed made thus:

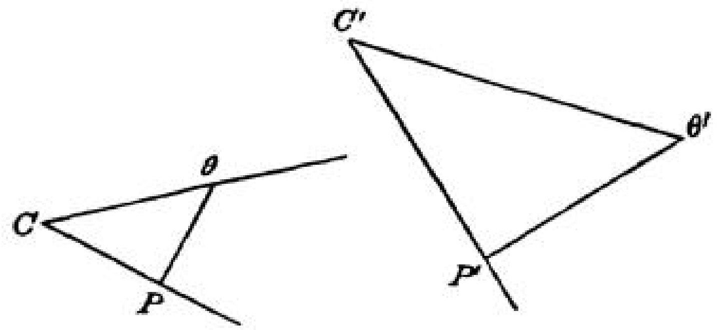
From any point draw rays in any directions; on each ray take two tracts in a fixed ratio; the ends form two similar figures. The fixed point is called *centre of similitude;* the fixed ratio, *ratio of similitude.*

Clearly the one figure may be thought as the other swollen or shrunk in like measure throughout.

By pushing or turning either figure the shape is not changed. When simply pushed, or when turned through a flat angle, corresponding tracts in the two figures keep ∥, and the figures keep *similarly placed.*

By the above construction of similar figures, space is doubled in thought: the space of the one figure, and the space of the other. To each point in one space *corresponds* a point in the other. If these spaces be now thought pushed or turned out from each other, corresponding points remain *centres of similarity* for the two figures and spaces. For, by similar triangles, corresponding tracts, tracts between corresponding points in pairs, are proportional and include equal angles.

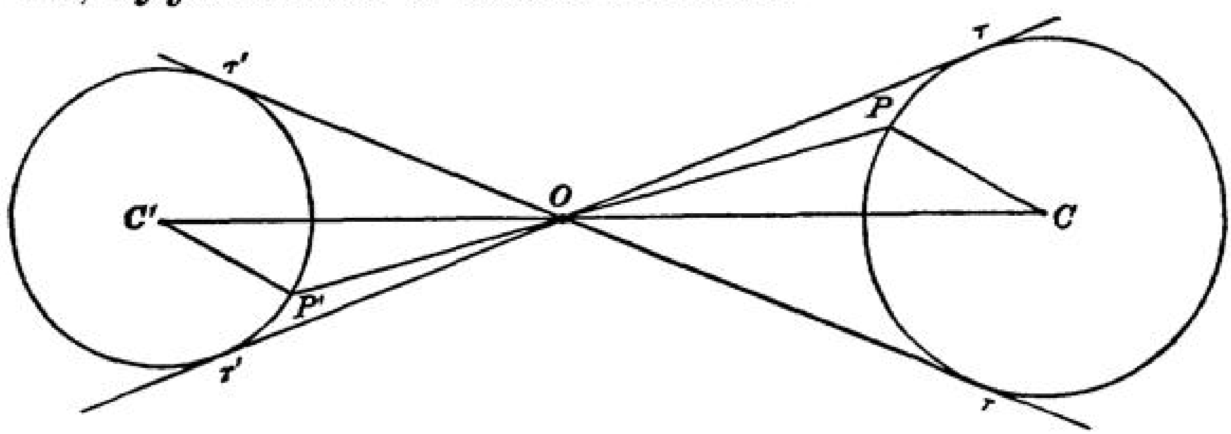
In what follows we shall keep the original construction, without pushing or turning, unless through a flat angle.



83. Clearly, tangents at corresponding points, being drawn through pairs of corresponding points, are ||.

To find the figure c′ similar to a circle c, radius r, ratio of similitude r : r′. Take any point O as a centre of similitude, lay off OC' so that $r:r' = OC:OC'$. Let P' correspond to P, then $OP:OP' = r:r' = OC:OC' = CP:C'P' = r:r'$.

Since CP is constant, so is $C'P'$; i.e., c' is a circle, radius r'; i.e., *a figure similar to a circle is a circle.*



Conversely, all circles are similar. For be c and c' any two circles, radii r and r'; cut the tract between their circles innerly respectively outerly in the ratio $r:r'$; then, by the above, the figure similar to c is a circle with centre C', radius r'; i.e., it is the circle c'.

Cor. 1. Any two circles in a plane have two centres of similitude: inner and outer, cutting the tract between their centres: innerly and outerly, in the ratio of their radii. This property is peculiar to the circle, since it alone of plane figures, being homogeneous, may be turned around in itself.

Cor. 2. The two pairs of common tangents to two circles cross in the inner resp. outer centre of similitude. Or thus:

A tangent to one of two circles through a centre of similitude is tangent to the other; for the radii to the points of touch are ||.

84. Be $C_n = (x - x_n)^2 + (y - y_n)^2 - r_n^2 = 0$ any circle, and let $I(C_1C_2)$ resp. $E(C_1C_2)$ denote the inner resp. outer centre of similitude of C_1 and C_2. Since $I(C_1C_2)$ resp. $E(C_1C_2)$ cuts the tract between the centres in the ratio of the radii, its Cds. are

$$x' = \frac{r_1x_2 + r_1x_1}{r_1 + r_2}, \quad y' = \frac{r_1y_2 + r_2y_1}{r_1 + r_2},$$

resp. $$x'' = \frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \quad y'' = \frac{r_1y_2 - r_2y_1}{r_1 - r_2}.$$

(x', y') resp. (x'', y'') is the pole of the chord of contact $T'T'$ resp. $T''T''$ of the inner resp. outer common tangents. Hence substituting for x', y' resp. x'', y'' in the Eq. of the polar, we get as the Eqs. of these four chords, after easy reduction:

$$(x_2 - x_1)(x - x_1) + (y_2 - y_1)(y - y_1) = r_1(r_1 \mp r_2),$$

resp. $$(x_1 - x_2)(x - x_2) + (y_1 - y_2)(y - y_2) = r_2(r_2 \mp r_1).$$

The centre-line is $(y_2 - y_1)(x - x_1) - (x_2 - x_1)(y - y_1) = 0$, whence we see the chords of contact are ⊥ to the centre-line.

85. Three circles, C_1, C_2, C_3, combining three ways in sets of two, have six centres of similitude: three inner, three outer. Form the Eq. of the RL. through $E(C_1C_2)$ and $E(C_2C_3)$, multiply it by $(r_1 - r_2)(r_2 - r_3)$, and divide it by r_2; the result is

$$\begin{vmatrix} r_1 & r_2 & r_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} x - \begin{vmatrix} r_1 & r_2 & r_3 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} y + \begin{vmatrix} r_1 & r_2 & r_3 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{vmatrix} = 0. \qquad (H')$$

To find the RL. through $E(C_2C_3)$ and $E(C_3C_1)$ permute the indices; this will not change H', it being symmetric as to the indices; i.e., the RLs. are the same; i.e., *the three outer centres of similitude*, $E(C_1C_2)$, $E(C_2C_3)$, $E(C_3C_1)$ *lie on a RL.* By changing the signs of the r's properly we show that

$$E(C_1C_2),\ I(C_2C_3),\ I(C_3C_1);$$
$$E(C_2C_3),\ I(C_3C_1),\ I(C_1C_2);$$
$$\text{resp. } E(C_3C_1),\ I(C_1C_2),\ I(C_2C_3)$$

lie on a RL. *These four RLs.* are named **axes of similitude** of the three circles: one *outer*, three *inner*.

COR. If two circles touch innerly resp. outerly, the point of touch is an outer resp. inner centre of similitude of the two; hence, from the above, if one circle touch two, the junction-line of the points of touch passes through a centre of similitude of the two: outer resp. inner according as the circles are touched alike (both outerly or both innerly) resp. not alike (one innerly, one outerly).

EXERCISE.

Find the centres and axes of

$$x^2 + y^2 = 16, \quad (x-5)^2 + y^2 = 81, \quad (x-y)^2 + (y-10)^2 = 4.$$

Draw the figure.

86. If a varying circle cut the circle $C_1 = 0$ under the angle a, by Art. 81, its radius R and its centre-Cds. satisfy the Eq. $R^2 - 2r_1R\cos a = C_1$. Treating R and a, or, what is the same, R^2 and $R\cos a$ as parameters, we may impose two more such conditions:

$$R^2 - 2r_2R\cos a = C_2, \qquad R^2 - 2r_3R\cos a = C_3;$$

eliminating the parameters from these three Eqs., we get a relation between the centre-Cds. and constants, i.e., we find the locus of the centre of the circle cutting the three circles $C_1=0$, $C_2=0$, $C_3=0$ under the same angle a. So we get

$$(r_1-r_3)(C_1-C_2)-(r_1-r_2)(C_1-C_3)=0,$$

a RL. through the intersection of $C_1-C_2=0$ *and* $C_1-C_3=0$, i.e., through the *power-centre* of $C_1=0$, $C_2=0$, $C_3=0$.

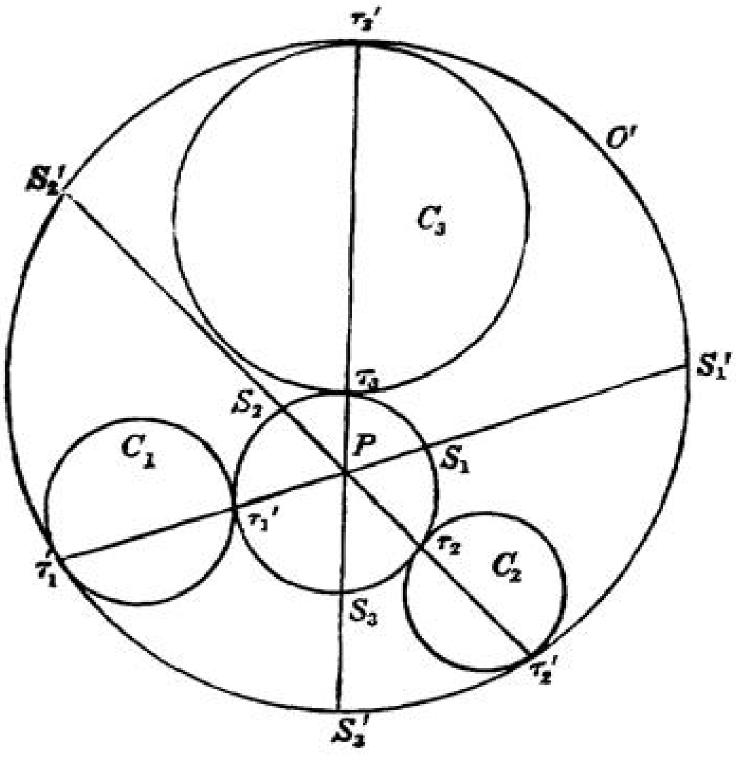
Writing for C_1-C_2, C_1-C_3 their values, we find the coefficients of x resp. y are the coefficients of y resp. $-x$ in the Eq. of an axis of similitude; i.e., the RL. is ⊥ to such an axis, and in fact the outer one. For thus far we have taken a as the *inner* angle between the circles, i.e., the angle between the radii to a section-point. To take a as the *outer* angle in case of either of the circles, it suffices: to change the sign of $\cos a$, since the inner and the outer angle are supplementary and cosines of supplementary angles are equal and unlike-signed; or to change the sign of r_1, r_2 resp. r_3. Taking all the angles as *outer* changes all the r's, which does not affect the Eq. of the RL.; changing *one* of the r's and leaving *two* unchanged is clearly tantamount to changing the *two* and leaving the *one* unchanged; this can be done in three ways; so we get *three other RLs. through the power-centre ⊥ each to an inner axis of similitude.* Hence the whole locus of the centre of a circle cutting three circles under the same (varying) angle is a pencil of *four RLs. through the power-centre, ⊥ each to an axis of similitude.*

87. If two circles touch *innerly*, their *inner* angle is 0; if *outerly*, it is 180°. Hence, the ⊥ from the power-centre on the *outer* axis of similitude contains the centres of *two* circles: one touching the three circles all innerly; the other, outerly: the r's are all + or all −. Changing the sign of one r we get a ⊥ on an *inner* axis of similitude containing the centres of *two* circles: one touching two circles innerly, the third outerly; the other touching the two outerly, the third innerly. Changing the sign of each r in turn we find in all eight circles touching each of three circles; a pair of centres lie on each ⊥ through the power-centre on an axis of similitude.

We might now determine another line on which the centre of a tangent circle must lie, by eliminating R between two Eqs. of condition, as $C_1 = R^2 + 2Rr_1$, $C_2 = R^2 + 2Rr_2$. But the line would not turn out to be a RL. or a circle, hence would not admit of elementary construction: with compasses and ruler. But the doctrines of poles and polars, power-centres and power-lines, centres and axes of similitude, enable us to solve the general Taction-Problem by use of the ruler alone.

88. Suppose the circle O resp. O' touches the given circles C_1, C_2, C_3 outerly resp. innerly.

(1) Then by Art. 85, Cor., the chords of contact T_1T_1', T_2T_2', T_3T_3' go through P, the inner centre of similitude of O and O'.



(2) Hence, $PT_1 : PS_1' = PS_1 : PT_1'$, $PT_1 \cdot PT_1' = PS_1 \cdot PS_1'$. But $PT_1 \cdot PS_1 \cdot PT_1' \cdot PS_1' = q^2 \cdot q'^2$, q^2 and q'^2 being powers of P as to O and O'. Hence, $PT_1 \cdot PT_1' = qq'$ (a constant for all directions) $= PT_2 \cdot PT_2' = PT_3 \cdot PT_3'$; hence P is the power-centre of C_1, C_2, C_3.

(N.B. T_1 and T_1' are **anti**-*correspondent* points of O and O'.)

(3) Since C_1 and C_2 touch O resp. O' each outerly resp. innerly, as in (1), the contact-chords T_1T_2 resp. $T_1'T_2'$ go through the same outer centre of similitude of C_1 and C_2, say K. Hence, as in (2),

$$KT_1 \cdot KT_2 = KT_1' \cdot KT_2';$$

i.e., the powers of K as to O and O' are equal; i.e., K is on the power-line of O and O'. So too likewise are the outer centres of similitude of C_2 and C_3, C_3 and C_1; i.e., the outer axis of similitude of C_1, C_2, C_3 is the power-line of O and O'.

(4) The power-lines of O and C_1 resp. O' and C_1, i.e., the tangents at T_1 resp. T_1', meet say at X_1, which is then the pole of the contact-chord T_1T_1' as to C_1; but by Art. 77 the power-line of O and O' also goes through X_1, the power-centre of O, O', C_1; i.e., through the pole of chord T_1T_1' as to C_1; hence, by Art. 69 the chord T_1T_1' goes through the pole of the power-line as to C_1.

Likewise the chord T_2T_2' resp. T_3T_3' goes through the pole of the same power-line as to C_2 resp. C_3.

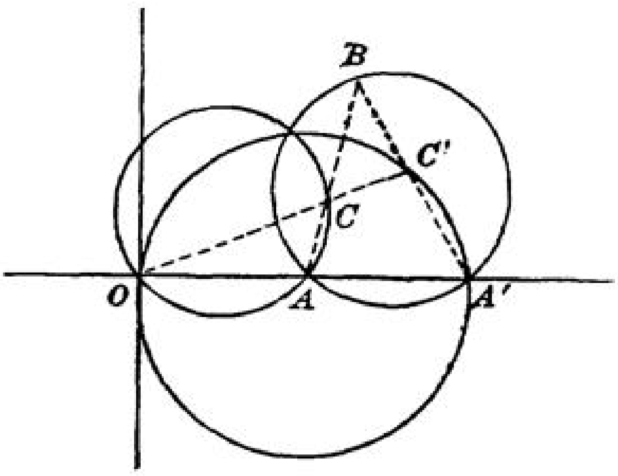
If, of the two circles O and O', O touch C_1 and C_2 outerly resp. innerly and C_3 innerly resp. outerly, but O' touch C_1 and C_2 innerly resp. outerly and C_3 outerly resp. innerly, then the power-line of O and O' passes through the outer centre of similitude of C_1 and C_2, through the inner centres of similitude of C_2 and C_3, C_3 and C_1; i.e., it is the inner axis of similitude $E_3I_1I_2$; the other relations hold unchanged. Hence the following rule:

Determine the power-centre and axes of similitude of the three given circles; determine the pole of each axis as to each circle: each three RLs. through the power-centre and the three poles of each axis cut the three circles in contact-points of two tangent circles.

Remark. This classic problem, in which the geometry of the circle seems to culminate, was proposed and solved by Apollonius of Pergæ (B.C. 220). His solution was lost, but was restored by Vieta (†1603). The first solution of the analogous problem for space: to find a sphere touching 4 given spheres, was given by Fermat (†1665). Both solutions were indirect, reducing the problem to simpler and simpler problems. Gaultier (1813) and Gergonne (1814) first gave direct solutions of the first problem. In the above rule replace 3, circle, axis by 4, sphere, plane, to solve the problem for space. Note carefully on what the solution turns: on determining the chords of contact in the given circles (spheres) by two points: the power-centre and a pole of an axis (plane) of similitude.

Circular Loci.

89. 1. Given any △ cut by a transversal through a fixed point of the base; through the fixed point and the intersections of the transversal with each side and the adjacent vertex at the base are drawn circles; find the locus of their intersection.



Be ABA' the △. Take the fixed point O as origin, the base as one rectangular axis, say X-axis. Be $OA = a$, $OA' = a'$; then the sides AB, $A'B$ are

$$y = c\,(x - a),$$
$$y = c'(x - a');$$

the transversal is $y = sx$; it cuts the sides at

$$\left(\frac{ca}{c - s}, \frac{sca}{c - s}\right), \left(\frac{c'a'}{c' - s}, \frac{sc'a'}{c' - s}\right) \text{ or } C, C'.$$

The circles through O, A, C resp. O, A', C' are

$$x^2 + y^2 - ax - \frac{a(cs + 1)}{c - s}\,y = 0, \quad \text{resp.} \quad x^2 + y^2 - a'x - \frac{a'(c's + 1)}{c' - s}\,y = 0.$$

Eliminating s, we get the locus of their intersection,

$$\{c(x^2 + y^2 - ax) - ay\} : \{x^2 + y^2 - ax + cay\}$$
$$= \{c'(x^2 + y^2 - a'x) - a'y\} : \{x^2 + y^2 - a'x + c'a'y\};$$

or,
$$(c - c')\,\{(x^2 + y^2)^2 - (a + a')x(x^2 + y^2) + aa'(x^2 + y^2)\}$$
$$+ (1 + cc')(a' - a)\,y\,(x^2 + y^2) = 0;$$

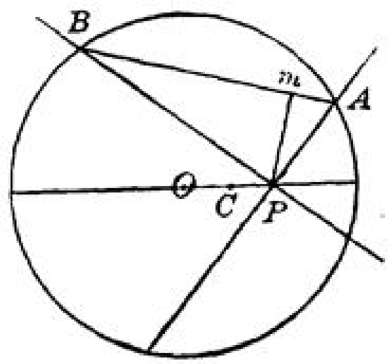
or, $$(x^2+y^2)\left\{x^2+y^2-(a+a')\,x-\frac{(1+cc')(a-a')}{c-c'}\,y+aa'\right\}=0.$$

This Eq. is the product of two: the first, of a point-circle, the origin; the second, of the circle circumscribing the $\triangle$.

2. A right angle turns about a fixed point; find the locus of the foot of the $\perp$ from the vertex on the chord of the intercept of its sides on a fixed circle.

The Eq. of the circle, referred to rectang. axes through the centre, is

$$x^2+y^2=r^2. \qquad (1)$$



Take the diameter through the fixed point P as X-axis. The chord is

$$y=sx+b. \qquad (2)$$

Eliminate y between (1) and (2), whence,

$$(1+s^2)\,x^2+2\,sbx+b^2-r^2=0. \qquad (3)$$

The roots of this Eq., x_1, x_2, are the x's of A, B; the y's are sx_1+b, sx_2+b. If $OP=c$, the direction-coefficients of the angle's sides PA, PB are

$$\frac{sx_1+b}{x_1-c},\qquad \frac{sx_2+b}{x_1-c};$$

since the angle is right, their product is -1; which, cleared of fractions, gives

$$(1+s^2)\,x_1x_2+(ab-c)(x_1+x_2)+b^2+c^2=0.$$

From (3) we have

$$x_1\,x_2=(b^2-r^2):(1+s^2),\qquad x_1+x_2=-2\,sb:(1+s^2)\cdot$$

$$\therefore\ (1+s^2)(c^2-r^2)+2\,b\,(sc+b)=0, \qquad (4)$$

an Eq. of condition between the parameters s and b.

The $\perp$ from P on the chord AB is

$$y=-\frac{1}{s}\,(x-c). \qquad (5)$$

Eliminating s and b, by means of (2), (4), (5), we get the locus of the intersection of (2) and (5), the locus sought:

$$\left\{y^2+(x-c)^2\right\}\cdot\left\{x^2+y^2-cx+\frac{c^2-r^2}{2}\right\}=0.$$

This breaks up into the Eq. of the point-circle P, and the circle about the mid-point of OP, radius $\sqrt{\frac{r^2}{2}-\frac{c^2}{4}}$.

Clearly P is not part of the locus sought; how, then, does it appear as part in the result? The pair of values, $x=c$, $y=0$, satisfies (5) for all values, real and imaginary, of s and b; then, from (2), $0=sc+b$; hence, from (4), for $c \lessgtr r$, $(1+s^2)=0$, $s=\pm i$.

Now the problem as proposed implied only real values of s and b, but the analytic statement held not only for real, but also for imaginary, values; i.e., for the problem in question and for more; accordingly, the result yields the locus sought, for real values of s and b, and another locus not sought, for imaginary values of s and b.

3. Find the locus of the foot of the ⊥ from the centre on the chord.

4. Find the locus of the intersection of tangents at the ends of the chord.

Be (x', y') the intersection; then $x'x+y'y=r^2$ is the chord. Combine this with $x^2+y^2=r^2$; the pairs of roots so obtained (x_1, y_1), (x_2, y_2) picture the ends A, B of the chord; the coefficients of direction of PA, PB are

$$\frac{y_1}{x_1-c}, \quad \frac{y_2}{x_2-c}, \text{ and their product is } -1.$$

Hence, $$(x'^2+y'^2)(r^2-c^2)+2r^2cx'-2r^4=0;$$

or, dropping the primes,

$$\left\{x+\frac{r^2c}{r^2-c^2}\right\}^2+y^2=\frac{r^4(2r^2-c^2)}{(r^2-c^2)^2}.$$

If R be the radius of this circle, d the distance between the centres,

then $$(R^2-d^2)^2=2r^2(R^2+d^2),$$

the condition that a quadrilateral inscribed in one circle may be circumscribed about another, any tangent to the inner being taken as a side.

5. Find the locus of a point the feet of perpendiculars from which, on the sides of a △, lie on a RL.

Be $x\cos\alpha_1+y\sin\alpha_1-p_1=0=N_1$, $N_2=0$, $N_3=0$ the sides of the △; (x_1, y_1), (x_2, y_2), (x_3, y_3) the feet of the ⊥s; (x, y) the point. Then are N_1, N_2, N_3 the lengths of these ⊥s. Their projections on the axes are

$$x-x_1=N_1\cos\alpha_1, \quad y-y_1=N_1\sin\alpha_1,$$

and so with indices 2, 3; hence,

$$x_k=x-N_k\cos\alpha_k, \quad y_k=y-N_k\sin\alpha_k,$$

for $k=1, 2, 3$.

The feet (x_1, y_1), (x_2, y_2), (x_3, y_3) lie on a RL. when, and only when,

$$y_2 - y_1 : x_2 - x_1 = y_3 - y_1 : x_3 - x_1;$$

or, after reduction and substitution,

$$N_1N_2 \sin(\alpha_1 - \alpha_2) + N_2N_3 \sin(\alpha_2 - \alpha_3) + N_3N_1 \sin(\alpha_3 - \alpha_1) = 0. \qquad (1)$$

N_1, N_2, N_3 are linear in x, y; hence, (1) is quadratic in x, y. The first term yields as coefficients of x^2, y^2, resp. xy,

$$\cos\alpha_1 \cos\alpha_2 \sin(\alpha_1 - \alpha_2),$$
$$\sin\alpha_1 \sin\alpha_2 \sin(\alpha_1 - \alpha_2),$$
$$\text{resp. } \sin(\alpha_1 + \alpha_2)\sin(\alpha_1 - \alpha_2).$$

The difference of the first two is

$$\cos(\alpha_1 + \alpha_2)\sin(\alpha_1 - \alpha_2),$$

or,
$$\tfrac{1}{2}(\sin 2\alpha_1 - \sin 2\alpha_2).$$

The third is
$$-\tfrac{1}{2}(\cos 2\alpha_1 - \cos 2\alpha_2).$$

Permuting the indices to get the contributions of the other terms, and summing, we see that the difference of the coefficients of x^2 and y^2, as well as the coefficient of xy, vanishes; i.e., the locus is a circle; the Eq. is also satisfied by putting any two of the N's $=0$; i.e., the circle goes through the vertices.

Hence, the feet of ⊥s on the sides of a △ from any point of the circumscribed circle, and from no other, lie on a RL.

This problem deserves further notice. Suppose the RLs. $N_1 = 0$, $N_2 = 0$, $N_3 = 0$ tangent to a circle O, radius r, centre at origin; then, $p_1 = p_2 = p_3 = r$. Developing (1), we have

$$M(x^2 + y^2) - Px - Qy + F = 0,$$

where
$$M = \sin(\alpha_1 - \alpha_2)\sin(\alpha_2 - \alpha_3)\sin(\alpha_3 - \alpha_1)$$

If R be the radius of this circle, D the distance between the centres,

$$R^2 = (P^2 + Q^2) : 4M^2 - \frac{F}{M},$$
$$D^2 = (P^2 + Q^2) : 4M^2,$$

whence,
$$D^2 - R^2 = \frac{F}{M}.$$

If A_1, A_2, A_3 be the angles of the △, then

$$\alpha_2 - \alpha_1 = \pi - A_3, \quad \alpha_3 - \alpha_2 = \pi - A_1, \quad \alpha_1 - \alpha_3 = \pi - A_2;$$

hence, if $S =$ area of △, $M = -\sin A_1 . \sin A_2 . \sin A_3 = -S : 2R^2$;

and from (2),
$$F = r^2(\sin A_1 + \sin A_2 + \sin A_3).$$

If s_1, s_2, s_3 be sides of the $\triangle$,

$$2S = r(s_1 + s_2 + s_3) = 2Rr(\sin A_1 + \sin A_2 + \sin A_3);$$

whence, $F = rS : R$, and hence, $F : M = -2Rr$;

$$\therefore D^2 = R^2 - 2Rr. \qquad (I)$$

Such, then, is the relation connecting the *radii* and *distance between the centres* of the inscribed and circumscribed circles of a $\triangle$. Now, holding the circles fixed, let us see how we can vary the $\triangle$.

Taking the one centre as origin, the other on the X-axis, we have

$$Q = 0, \quad R^2 = \frac{P^2}{4M^2} - \frac{F}{M}, \quad D^2 = \frac{P^2}{4M^2}.$$

By virtue of (I), these *three* relations are satisfied if these *two* are:

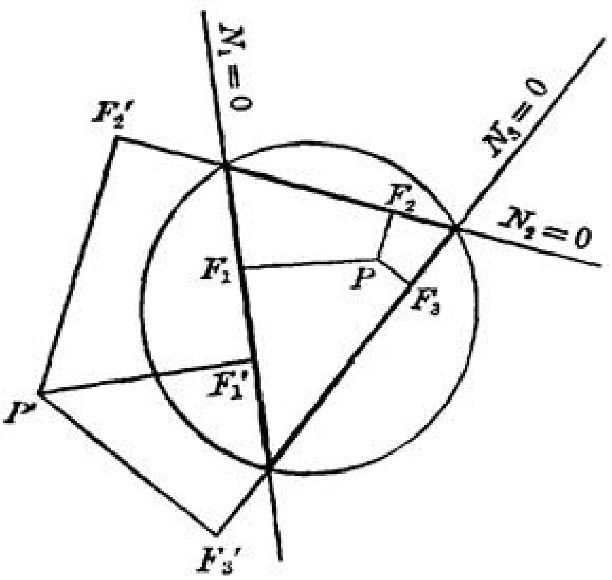
$$Q = 0, \quad F : M = -2Rr.$$

Choosing one of the angles α_1, α_2, α_3 at pleasure, we can still find values of the other two to satisfy these two equations.

Hence, *When the relation* $D^2 = R^2 - 2Rr$ *holds between the radii of two circles and the distance between their centres, a $\triangle$ can be drawn in the one about the other, the direction of one side being taken at pleasure.*

Theorems analogous to these two for triangles and quadrilaterals hold for polygons generally.

We got (1) by imposing the condition that the feet of the $\perp$s lie on a RL. Let us see how it expresses this condition. Suppose the origin inside of the $\triangle$ and $\alpha_1 < \alpha_2 < \alpha_3$. Be $P(x, y)$ any point within the $\triangle$, and join the feet of the $\perp$s from it on the sides of the $\triangle$, to form a $\triangle$ F_1, F_2, F_3. Then N_1, N_2, N_3 are the lengths of these $\perp$s, all have the same sign $-$, and are like-signed with the $\perp$s from the origin inclined α_1, α_2, α_3, as both P and the origin are within the $\triangle$. Hence the terms of the left side of (1) are in order the double areas of the $\triangle$s F_1PF_2, F_2PF_3, F_3PF_1, the whole left side is the double area of $F_1F_2F_3$.



If $P'(x, y)$ be without the $\triangle$, then one of the $\perp$s, say N_1, becomes $+$, and the left side of (1) is the difference between the double area of the $\triangle$ $F_2'P'F_3'$ and the sum of the double areas of $F_1'P'F_2'$ and $F_3'P'F_1'$, i.e., again, the whole left side is the double area of $F_1'F_2'F_3'$. Now this double area is 0 when, and only when, F_1, F_2, F_3 are on a RL. Accordingly we may generalize our problem by requiring that the area of the $\triangle\, F_1F_2F_3$ be not 0 but some constant $\pm c^2$. Then the left side of (1) is this area doubled; also since (1) is the Eq. of a circle whose radius we call R, the left side can be written $K(d^2 - R^2)$, where K is constant and d^2 stands for the general expression, $x^2 + y^2 + 2gx + 2fy$. Hence

$$N_1N_2 \sin(\alpha_2 - \alpha_1) + N_2N_3 \sin(\alpha_3 - \alpha_2) + N_3N_1 \sin(\alpha_1 - \alpha_3)$$
$$= \pm 2c^2 = K(d^2 - R^2)$$

is the locus of a point the feet of $\perp$s from which on the sides $N_1 = 0$, $N_2 = 0$, $N_3 = 0$ of a $\triangle$ are vertices of a $\triangle$ of constant area c^2, + or −. The locus is two circles concentric to the circle circumscribing the given $\triangle$. The outer is always real; the inner, only when $c^2 < KR^2$. The given area c^2 changes sign for $d = R$, i.e., as P goes through the circle.

6. Find the locus of a point, the sum of whose squared distances from n points, multiplied resp. by given constants, shall be a given constant.

7. Find the locus of the centre of a circle seen from two given points under given angles.

8. Find the locus of the centre of a circle that cuts two given circles at ends of diameters of each.

9. From a fixed point P are drawn tangents to a system of circles through two fixed points; find the locus of the intersection of the chord of contact with the diameter through P.

10. Be 1 2 3 1′ 2′ 3′ a regular hexagon; draw 1 3, 1 3′, also any RL. through the centre, cutting 1 3, 1 3′ at 4, 4′; find the locus of the intersection of 2 4 and 2′ 4′.

11. Find the locus of a point whose polars as to three given circles meet in a point.

12. A constant angle turns about its vertex fixed on the bisector of a fixed angle; find where the $\perp$ from the vertex on the junction-line of the intersection of the sides of the angles meets it.

13. Find the locus of a point from which two given circles seem of like size.

14. Find the locus of a point whence two consecutive tracts LM, MN of a RL. seem of like size.

15. Find the locus of the mass-centre of a $\triangle$ inscribed in a given circle, on a given chord as base.

16. Under the same conditions as in (15) find the locus of the orthocentre and of the centre of sides of the △.

17. Of two related poles, as to a given circle, one glides along a RL.; how does the other glide?

18. How does one of two related poles as to $x^2 + y^2 = r^2$ glide when the other glides along $(x - a)^2 + y^2 = R^2$?

19. Find the locus of the mid-point of a chord of a given circle, which subtends a right angle at a given point.

20. Find the locus of the foot of the ⊥ from the origin on a chord of a given circle, which subtends a right angle at the origin.

21. Two variable circles touch each other and two fixed circles; find the locus of their point of touch.

22. Find the locus of a point whose polars as to three fixed circles meet in a point.

CHAPTER IV.

GENERAL PROPERTIES OF CONICS.

90. The general Eq. of curves of second degree, called **Conics** (Art. 152), is

$$kx^2 + 2\,hxy + jy^2 + 2\,gx + 2\,fy + c = 0,$$

or $\quad F(x, y\,;\, x, y) = 0.$

Among many ways of treating conics that seems most natural which proves itself the best in the study of Quadrics (surfaces of second degree), namely, to develop the relations of the locus to the RL. Let the student recall that

The Eq. of the polar as to $F(x, y\,;\, x, y) = 0$ of the pole (x_1, y_1) is

$$F(x_1, y_1\,;\, x, y) = 0, \quad \text{or} \quad F(x, y\,;\, x_1, y_1) = 0 \qquad \text{(Art. 69)},$$

or $\quad (kx_1 + hy_1 + g)x + (hx_1 + jy_1 + f)y + gx_1 + fy_1 + c = 0. \quad \text{(J)}$

If the pole be on the curve, the polar is a tangent at the pole.

The Eq. of the pair of tangents to $F(x, y\,;\, x, y) = 0$ through (x_1, y_1) is

$$F(x_1, y_1\,;\, x_1, y_1) \cdot F(x, y\,;\, x, y) = \{F(x_1, y_1\,;\, x, y)\}^2 \text{ (Art. 72).}$$

By passing to ∥ axes through a new origin, $O'(x', y')$, k, h, j are not changed, but g, f, c are changed into

$$g' = kx' + hy' + g,$$
$$f' = hx' + jy' + f,$$
$$c' = F(x', y'\,;\, x', y') \qquad \text{(Art. 51).}$$

If $\Delta \equiv \begin{vmatrix} k & h & g \\ h & j & f \\ g & f & c \end{vmatrix} = 0,$ the conic breaks up into two RLs.

If, also, $C \equiv kj - h^2 = 0$, the RLs. are ∥.

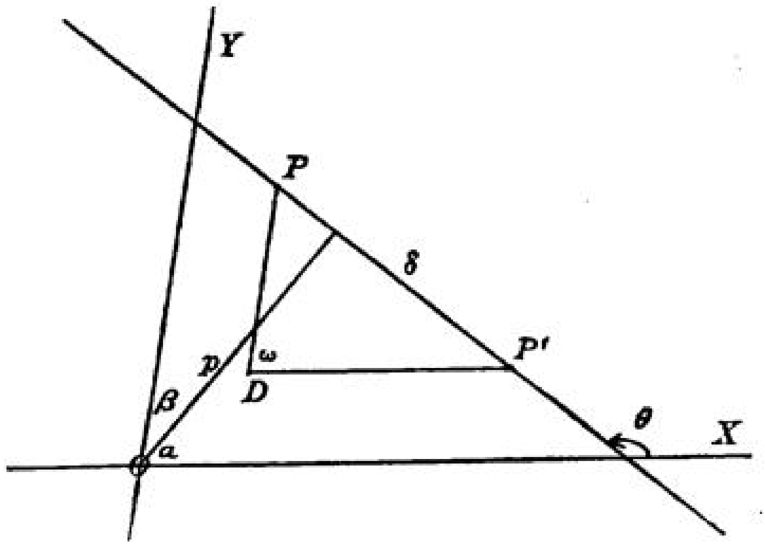
91. If (x', y') and (x, y) be a fixed and a variable point on a RL. sloped θ to the X-axis, the $\perp$ to which is sloped α resp. β to the X- resp. Y-axis, and δ be the distance between the points, then either of these two equivalent sets of relations states the Law of Sines of the $\triangle\ PDP'$:

$$\frac{x-x'}{\sin\overline{\omega-\theta}}=\frac{y-y'}{\sin\theta}=\frac{\delta}{\sin\omega};\quad \frac{x-x'}{\cos\beta}=\frac{y-y'}{\cos\alpha}=\frac{\delta}{\sin\omega}.$$

Put $\quad q'\equiv\dfrac{\sin\theta}{\sin\omega}=\dfrac{\cos\alpha}{\sin\omega},\quad q\equiv\dfrac{\sin\overline{\omega-\theta}}{\sin\omega}=\dfrac{\cos\beta}{\sin\omega};$

whence $\quad s\equiv\dfrac{\sin\theta}{\sin\overline{\omega-\theta}}=\dfrac{q'}{q}.$

Then $\quad x=x'+q\delta,\qquad y=y'+q'\delta.$



To find the distances from (x', y') at which the RL. meets the conic, substitute in $\quad F(x, y;\ x, y)=0,\quad$ as in Art. 51; hence

$$(kq^2+2hqq'+jq'^2)\delta^2+2\{(kx'+hy'+g)q + (hx'+jy'+f)q'\}\delta+F(x',y';\ x',y')=0.$$

A geometric interpretation of this Eq., according to Art. 60, lays bare the form and general properties of the conic.

92. The roots of this quadratic, δ_1, δ_2, are counter (equal and unlike-signed), i.e., (x', y') is the *mid-point* of a chord of the conic, directed by θ, when, and only when,

$$(kx'+hy'+g)q+(hx'+jy'+f)q'=0,$$

or $$(kx'+hy'+g)+s(hx'+jy'+f)=0. \qquad (K)$$

For θ constant, s, q, and q' are constant, and the Cds. (x', y') of the *mid-point* of *any* chord directed by θ are connected by an Eq. of first degree; i.e., *the mid-points of all* || *chords of a conic lie on a RL.* Such a RL. is named a **Diameter**.

By changing θ we change the direction of the || chords, change s, and change the diameter; s, then, is the parameter of the system of diameters, and since it enters (K) linearly, *all diameters pass through a point*, called the **centre** of the conic.

This centre is the intersection of the two diameters

$$kx'+hy'+g=0 \quad \text{and} \quad hy'+jy'+f=0;$$

i.e., *the point* $(G:C,\ F:C)$. It is in finity or in ∞ according as $C \gtrless 0$ or $C=0$. Hence conics are named conveniently, but not quite correctly, **centric** and **non-centric**, according as the centre lies in finity or not in finity.

93. We have got the notion of centre from that of diameter, but we may get this from that, thus:

The coefficients of x and y in Eq. (J) of the polar of (x_1, y_1) vanish when $kx_1+hy_1+g=0$ and $hx_1+jy_1+f=0$, i.e., when the pole is the point $(G:C,\ F:C)$; but when the coefficients vanish, the intercepts on the axes are both ∞, i.e., *the RL. lies wholly in* ∞.* Hence $(G:C,\ F:C)$ is the pole of the polar at ∞. Hence the polar of every point at ∞ goes through $(G:C,\ F:C)$. Now a pole and any point on its polar are a pair of points to which the section-points with the referee of a RL. through the pole and point on the polar are an harmonic pair; and since one point of the first pair is at ∞, the other halves the tract between the second pair (Art. 41); i.e., *the point* $(G:C,\ F:C)$ *halves every chord of the referee* (conic) *through it.* Such a point is named **centre**.

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* See Note, page 196.

Now take any point on a RL. or polar through this centre; by Art. 71 it is the fourth harmonic to the pole at ∞, and hence halves the tract between the other pair, namely, the section-points with the referee (conic) of the RL. through it and the pole. But all such RLs. are ∥, since, as the point glides along the polar, they turn about the same point at ∞, the pole of that polar; hence, too, they are all *conjugate* to that polar and no other RLs. are; hence, *a RL. through the centre of a conic halves a system of ∥ chords which are conjugate to it as a polar.* Such a RL. is called a **Diameter** *conjugate* to the chords it halves.

Among all the chords conjugate to any diameter D there is one and only one through the centre, which is accordingly D's *conjugate diameter* D'. The chords which D' halves are ∥ to D. For D passes through the pole of D' at ∞, and hence all ∥'s to D pass through that pole at ∞; hence all chords ∥ to D are conjugate to D', and hence halved by D'. Hence, *conjugate diameters, D and D', of a conic halve each all chords ∥ to the other.* This, indeed, is clear from the fact that the conjugate relation is mutual (Art. 69).

94. From (K) we see that the direction-coefficient s' of the diameter halving chords whose direction-coefficient is s is

$$s' = -\frac{k+hs}{h+js}; \quad \text{whence} \quad s = -\frac{k+hs'}{h+js'}.$$

If $$C \equiv hj - h^2 = 0,$$

then $$s' = -\frac{\sqrt{k}}{\sqrt{j}}, \qquad s = -\frac{\sqrt{k}(\sqrt{k}+\sqrt{j}\cdot s')}{\sqrt{j}\,(\sqrt{k}+\sqrt{j}\cdot s')}.$$

Hence, s' is constant, i.e., *diameters of a non-centric conic are* ∥. We may not cancel $\sqrt{k}+\sqrt{j}\cdot s'$ in the numerator and denominator of s, since it is 0, and to divide by 0 has no sense; but s takes the undetermined form $\frac{0}{0}$. To find what this means, put s' or its value for s in (K); so we get the Eq. of the diameter conjugate to the chords directed by s, i.e., the ∥ diameters. Reducing, we get $0x + 0y + g\sqrt{j} + f\sqrt{k} = 0$; but this, by Art. 92, is the Eq. of a RL. at ∞. Now this RL. at ∞ goes through the centre at ∞, and so halves all chords through that centre, i.e., halves all the ∥ diameters, i.e., is conjugate to them all; and this it does whatever its direction may be.

Hence, the common conjugate to all diameters of a non-centric conic is the RL. at ∞, and may be thought ∥ to them all.

95. For $\alpha = 90°$, q' is 0, and the diameter is

$$kx' + hy' + g = 0;$$

this, then, is the diameter halving chords ∥ to the X-axis. So, too, $hx' + j'y' + f = 0$ is the diameter of chords ∥ to the Y-axis. These diameters are themselves ∥ to the Y- resp. X-axis when and only when $h = 0$; but then they are conjugate, as each halves chords ∥ to the other; hence, *the condition necessary and sufficient that the axes be ∥ to a pair of conjugate diameters is* $h = 0$, i.e., *the term in* xy *must vanish.*

If the centre be taken as origin (which can be done always and only in centric conics), the new coefficients of x and y, $kx_1 + hy_1 + g$ and $hx_1 + jy_1 + f$, vanish, and the central Eq. becomes

$$kx^2 + 2hxy + jy^2 + c' = 0.$$

If, besides, a pair of conjugate diameters be taken as axes, the term in xy vanishes, and the Eq. takes the form

$$k'x^2 + j'y^2 + c' = 0.$$

This Eq. is a pure quadratic both in x and in y: to any value of either correspond two counter values of the other, each axis halving all chords ∥ to the other.

96. In general, diameters are *oblique* to their conjugate chords; are they ever ⊥? Choose rectang. axes; then s and s' become $\tan\theta$ and $\tan\theta'$, and

$$\tan\theta' = -\frac{k + h\tan\theta}{h + j\tan\theta}.$$

When chords and diameter are ⊥, $\tan\theta \cdot \tan\theta' = -1$; or, on reduction

$$h\,\overline{\tan\theta}^2 + \overline{k-j}\,\tan\theta - h = 0.$$

This quadratic in $\tan\theta$ has two roots: $\tan\theta_1$, $\tan\theta_2$, both always real. They yield each an ∞ of values of θ, but as they differ among themselves only by some multiple of π, the two ∞'s determine but two different directions of chords ⊥ to their diameters; also, since $\tan\theta_1 \tan\theta_2 = -1$, these directions are ⊥ to each other: hence, if either be thought as the direction of the chords, the other will be the direction of their diameter. Again, by Art. 52, these two ⊥ directions are the ones that halve the angles between the directions fixed by the pair of RLs. $kx^2 + 2\,hxy + jy^2 = 0$. Hence, *there is one and only one pair of ⊥ conjugate diameters:* the pair halving the angles between the *Asymptotes* (see Art. 97). They are named **Axes** of the conic.

N.B. Of course, in the non-centric conic only one diameter ⊥ to its chords is in finity; it is called the **Axis** of the conic.

97. If the coefficient of the second power of δ be 0, one root, δ_1, of the Eq. is ∞; i.e., one distance from (x', y') to the conic in a direction fixed by

$$kq^2 + 2\,hqq' + jq'^2 = 0, \qquad \text{or} \qquad k + 2\,hs + js^2 = 0$$

is ∞. This Eq. is quadratic in s; hence there are *two* such directions, which are real and separate, real and coincident, or imaginary according as $h^2 - kj$(or $-C$) is > 0, $= 0$, or < 0. The conic is named accordingly **Hyperbola** (excess), **Parabola** (likeness), or **Ellipse** (lack). Denote them by ***H***, ***P***, ***E***. ***H*** has points at ∞ in *two* directions, ***P*** in *one*, ***E*** in *none* (real).

The co-factor C is called the **criterion** of the conic. The direction in which lies the point at ∞ in ***P*** is fixed by the value of s: $s = \sqrt{k} : \sqrt{j}$, since $kj - h^2 = 0$; but this is the direction-coefficient of the ∥ diameters; hence *all diameters of* *the non-centric* ***P*** *meet it at* ∞, and hence can meet in only one point in finity.

98. If the coefficients of both powers of δ vanish, then both roots, δ_1, δ_2, are ∞; the coefficient of δ vanishes when and only when both

$$kx' + hy' + g = 0 \quad \text{and} \quad hx' + jy' + f = 0;$$

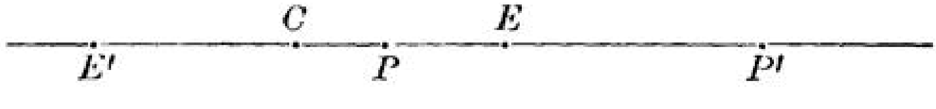
i.e., only when the origin is at the centre; i.e., in ***H*** and ***E***, *not* in ***P***. Hence, *two RLs.* drawn through the centre in directions fixed by $k + 2hs + js^2 = 0$ *meet the centric conic each in two points at* ∞. Now the points at ∞ on a RL. are consecutive; hence, these RLs. meet the centric conic at two consecutive points at ∞; i.e., *they touch it at* ∞.

These RLs. through the centre tangent to the centric conic at ∞ are named **Asymptotes**; they are real in ***H***, imaginary in ***E***. Their directions may be named *asymptotic*. All RLs. drawn in asymptotic directions, except the asymptotes, meet the conic in one finite, and one infinite, point; both points are real in ***H***, imaginary in ***E***.

In the non-centric conic, ***P***, the *one diametral* direction represents *two coincident asymptotic* directions; all diameters of ***P*** meet the curve in one finite, and one infinite, point.

99. If E, E', C be ends and centre of a diameter, P any pole on it, P' the section of the diameter with P's polar, then, by Art. 71, E, P, E', P' form an harmonic range;

$$\therefore \frac{EP \cdot E'P'}{PE' \cdot P'E} = -1, \quad \text{or} \quad EP : PE' = EP' : E'P'.$$



On compounding and dividing, results

$$CP : CE = CE : CP';$$

i.e., *The geometric mean of the central distances of any pole and its polar, measured on any diameter, is half that diameter.*

In ***P*** the centre C and one end of the diameter, say E', retire to ∞; hence E' halves PP' outerly, and hence *E halves PP' innerly.*

By definition the poles of a system of $\|$ chords lie on the diameter, D, conjugate to the chords, halving them; hence *the tangents at the ends of any one of these chords meet on the diameter, D, in the pole of that chord;* if the chord be a diameter, D', its pole is the point at ∞ on D, and the tangents through its ends are accordingly $\|$ to each other and to D; i.e., *tangents at the ends of a diameter are $\|$ to its conjugate.*

In ***P*** this conjugate is at ∞, but the *tangent* is still *$\|$ to the diameter's conjugate chords.*

100. We have reduced (Art. 94) the Eq. of centric conics, ***H*** and ***E***; to reduce that of the non-centric, ***P***, take as X-axis any diameter, as Y-axis the tangent through its end. Then the absolute vanishes, the origin being on the curve; the Eq. becomes a pure quadratic in y, since to any value of x must correspond two counter values of y, the chords $\|$ to the Y-axis being halved by the X-axis; the term in x^2 vanishes, since, for any value of y, one of the x-roots of the Eq. must be ∞, one finite, the $\|$'s to the X-axis, i.e., the diameters, meeting the curve in one point in finity, one in ∞; there remain only the terms in x and y^2, which may be written conveniently thus: $\boldsymbol{y^2 = 4q'x}$, the Eq. of the ***P*** referred to a diameter and the tangent at its end.

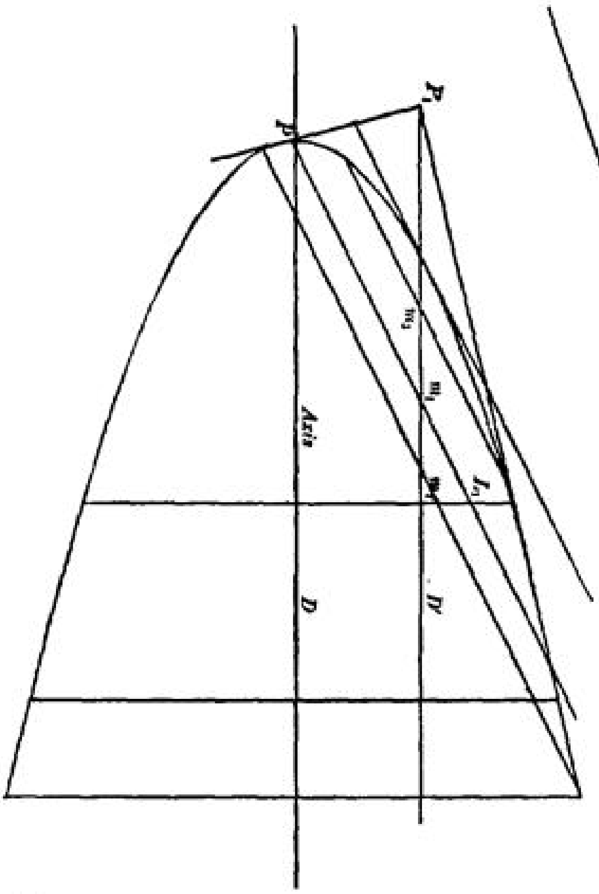
The figures on page 121 illustrate fully the foregoing articles.

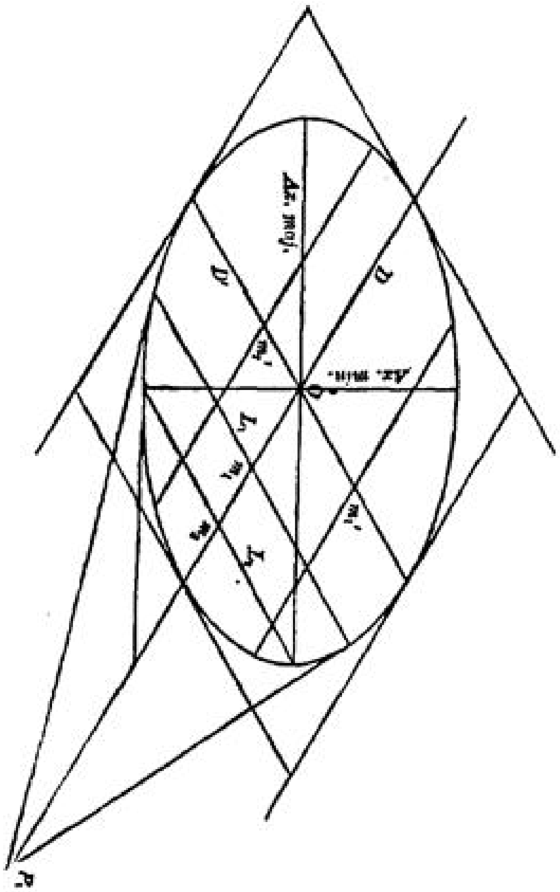
101. Let us resume the study of the quadratic in δ. The coefficient of δ^2 contains k, h, j, θ, and may be written

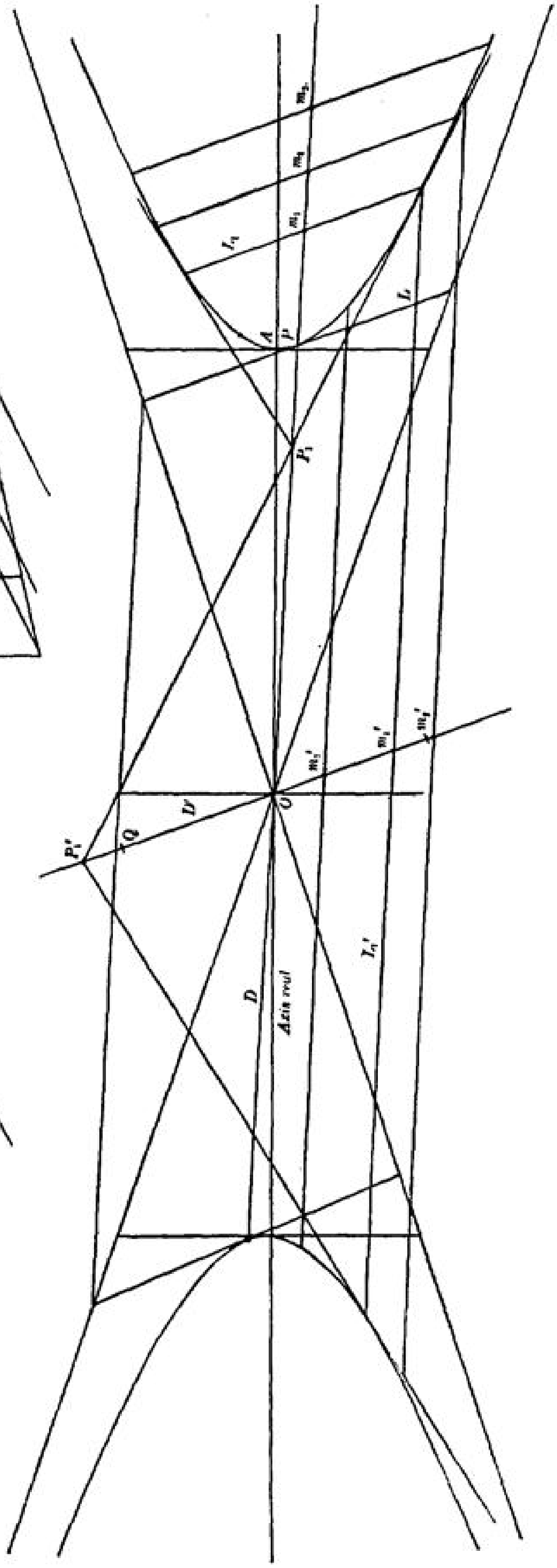
$$\phi(k, h, j; \theta);$$

then the product of the roots δ_1, δ_2, i.e., of the distances of (x', y') from the conic in the direction θ', is

$$F(x', y'; x', y') : \phi(k, h, j; \theta')$$


Axis
D
D'
P_1


Ax. maj.
Ax. min.
D
D'
O


Axis real
D
D'
O
Q
A
P_1

The product of the distances of (x'', y'') in the same direction is

$$F(x'', y''; x'', y'') : \phi(k, h, j; \theta').$$

The ratio of these products

$$F(x', y'; x', y') : F(x'', y''; x'', y'')$$

is independent of θ', i.e., the same for all directions; i.e., *the ratio of the products of the distances of any two points from a conic is constant for all* || *directions of the distances.*

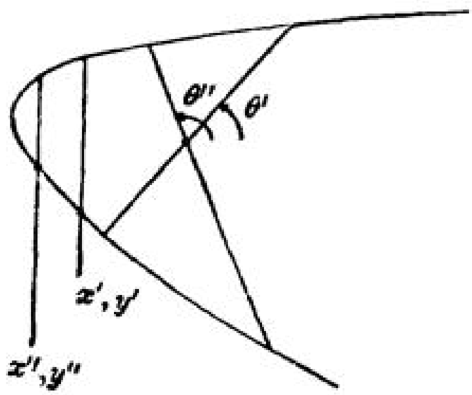
By taking two fixed directions θ', θ'' and one arbitrary point (x', y'), instead of two fixed points (x', y'), (x'', y'') and one arbitrary direction θ', we get as ratio of the product of the distances

$$\phi(k, h, j; \theta') : \phi(k, h, j; \theta''),$$

— a result independent of the point (x', y'), the same for all points; i.e., *the ratio of the product of the distances of a point from a conic, measured in two fixed directions, is constant for all points.*

The interest of these theorems lies mainly in the special cases:

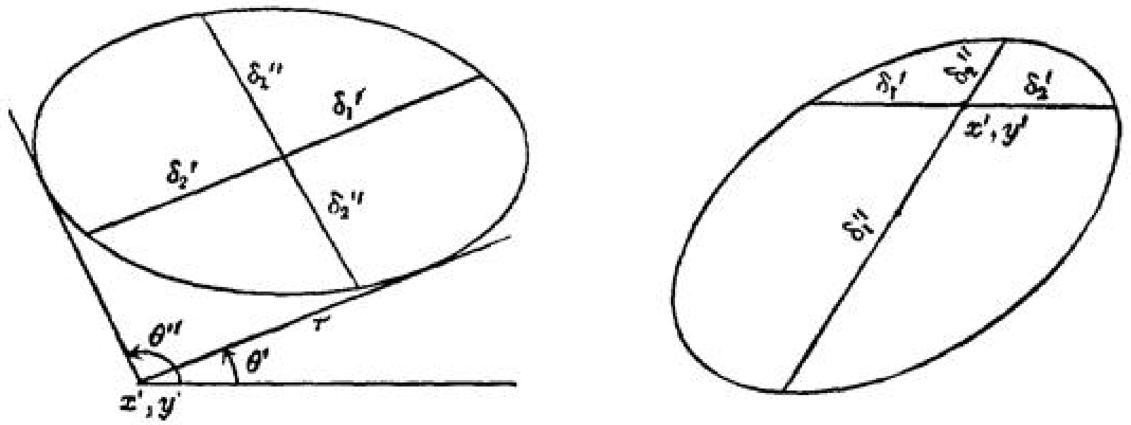
(1) Take the centre as the point; then the two distances in any direction are equal, being halves of a diameter; hence *the ratio of the products of the distances from any point to a conic is the ratio of the squared diameters* || *to the distances.*



(2) Take the directions tangent to the conic; then the distances are again equal; taking the second root of the ratio of

the products, we see that *the ratio of two tangent-lengths from a point to a conic equals the ratio of the ∥ diameters.*

(3) Take as directions those of the diameter through the point and its conjugate chord; then the distances on the chord are equal, and those on the diameter are its segments; hence *the square of any chord varies as the product of the segments into which it cuts its conjugate diameter.* See the figures.



102. By Art. 51 k, h, j are not changed by a change of origin; to find how they are changed by a change of axial directions, we might use the general formulae of transformation (Art. 21); much neater, however, is this method of Boole:

The transformation formulae being homogeneous in Cds., in passing from axes X, Y to axes X', Y', inclined ω resp. ω', the expression $kx^2 + 2hxy + jy^2$ changes into

$$k'x'^2 + 2h'x'y' + j'y'^2,$$

so that $$kx^2 + 2hxy + jy^2 = k'x'^2 + 2h'x'y' + j'y'^2.$$

Also $$x^2 + 2xy\cos\omega + y^2 = x'^2 + 2x'y'\cos\omega' + y'^2,$$

since each is the squared distance of the same point (x, y) (x', y') from the common origin. Add this Eq., multiplied by an arbitrary μ, to the first; there results

$$(k+\mu)x^2 + 2(h+\mu\cos\omega)xy + (j+\mu)y^2$$
$$= (k'+\mu)x'^2 + 2(h'+\mu\cos\omega')x'y' + (j'+\mu)y'^2.$$

Each side of this Eq., equated to 0, represents the same locus: a pair of RLs. through the origin (Art. 44); if μ be chosen so that the RLs. fall together, each side becomes a perfect square; i.e., the same values of μ make both sides perfect squares; i.e., the roots, μ_1, μ_2, of the two Eqs.:

$$(k+\mu)(j+\mu)=(h+\mu\cos\omega)^2$$

and $$(k'+\mu)(j'+\mu)=(h'+\mu\cos\omega')^2$$

are the same; i.e., corresponding ratios of the coefficients of the powers of μ in the two Eqs. are equal; i.e.,

$$(k+j-2h\cos\omega):\overline{\sin\omega}^2=(k'+j'-2h'\cos\omega'):\overline{\sin\omega'}^2,$$

$$(kj-h^2):\overline{\sin\omega}^2=(k'j'-h'^2):\overline{\sin\omega'}^2;$$

i.e., *the ratios* $(k+j-2h\cos\omega):\overline{\sin\omega}^2$ and $(kj-h^2):\overline{\sin\omega}^2$ *are unchanged by any change of axes.*

Geometric Interpretation.

Suppose the central Eq. of a centric conic brought to the form:

$$kx^2+2hxy+jy^2=1;$$

then are $\frac{1}{\sqrt{k}}$, $\frac{1}{\sqrt{j}}$ the intercepts on the axes, and are half-diameters.

1. For $\omega=90$, from the above, $k+j$ is constant; i.e., *the sum of the squared reciprocals of two rectang. diameters is constant.*

2. For conjugate diameters taken as axes, $h=0$; hence $\frac{kj}{\overline{\sin\omega}^2}$ and $\frac{k+j}{\overline{\sin\omega}^2}$ are constant; hence their quotient $\frac{1}{k}+\frac{1}{j}$, or *the sum of two squared conjugate diameters, is constant.*

Also, by inverting and taking the second root, $\frac{\sin\omega}{\sqrt{k\cdot j}}$ is constant; i.e., *the area of the parallelogram of two conjugate half-diameters is constant.*

CHAPTER V.

SPECIAL PROPERTIES OF CONICS.

Centric Conics: Ellipse and Hyperbola.

103. The Eq. of the centric conic referred to conjugate diameters is

$$k'x^2 + j'y^2 + c' = 0 \qquad \text{(Art. 94)}.$$

Two general cases present themselves:

I. k' and j' *like-signed*, say both $+$; then the criterion $C \equiv kj - h^2 = k'j' - 0 > 0$; hence *the curve is an ellipse.* Under this head are two special cases:

(1) $c' < 0$; then the ellipse is *real*, denote it by ***E***.

(2) $c' > 0$; then the ellipse is *imaginary*, denote it by ***E'***.

For clearly no real values of x and y satisfy its Eq.

II. k' and j' *unlike-signed*, say $k'+$ and $j'-$; then the criterion $C \equiv kj - h^2 = k'j' - 0 < 0$; hence *the curve is an hyperbola.* Under this head are two special cases:

(1) $c' < 0$; then the hyperbola is *primary*, denote it by ***H***.

(2) $c' > 0$; then the hyperbola is *secondary*, denote it by ***H'***.

Now write $\frac{1}{a'^2}, \frac{1}{b'^2}$ for $-\frac{k'}{c'}, -\frac{j'}{c'}$; on observing signs there result these Eqs. of ***E***, ***E'***, ***H***, ***H'***, *referred to conjugate diameters:*

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1, \qquad \frac{x^2}{a'^2} + \frac{y^2}{b'^2} = -1,$$

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1, \qquad \frac{x^2}{a'^2} - \frac{y^2}{b'^2} = -1.$$

To denote that the pair of rectang. conjugate-diameters, or the axes of the conic, are taken as Cd. axes, *drop the primes from a and b.*

104. Thus far little reference has been made to figures, for the shapes of the curves were supposed unknown, and it was deemed important to illustrate how the properties of curves may be deduced while their forms are yet unknown. Reason far outruns imagination. We may reason correctly about forms we cannot imagine at all. But we may now find out the shapes and draw the figures of three of the above curves. Only the imaginary ***E'*** is unrepresentable in our plane.

Putting $y=0$ in the Eqs. of ***E***, ***H***, ***H'***, there results

$$x=\pm a, \quad x=\pm a, \quad x=\pm ia;$$

i.e., all three cut the X-axis on each side a from the origin (centre): ***E*** and ***H*** in real points, ***H'*** in imaginary points.

So, if $x=0$, then $y=\pm b$, $y=\pm ib$, $y=\pm b$; i.e., all three cut the Y-axis on each side b from the origin (centre): ***E*** and ***H'*** in real points, ***H*** in imaginary points.

It is common to assume $a>b$ in the ***E***; then $2a$ and $2b$ are called *axes major and minor* of the ***E***. $2a$ resp. $2b$ is the *real* (commonly called *transverse*) *axis* of ***H*** resp. ***H'***; $2ib$ resp. $2ia$ is the *imaginary axis* of ***H*** resp. ***H'***. The *real* axis $2b$ of ***H'*** is often called, though hardly properly, the *conjugate* axis of ***H***.

Plainly, like results hold when a and b are primed; i.e., when any pair of conjugate diameters are taken as Cd. axes: ***E*** cuts both in real points like-distant from the centre; ***H*** cuts only *one* of two conjugate diameters in real points, while ***H'*** cuts the *other*. Hence, while all the real ends of one system of diameters lie on ***H***, all the real ends of their conjugates lie on ***H'***; and conversely. Hence ***H'*** is commonly called the conjugate of ***H***; strictly each is the conjugate of the other.

Since the rectang. Eqs. are pure quadratics in both x and y, each curve is symmetric as to each of its axes.

Clearly a and b are the greatest values of x and y in ***E***; a is the least value of x in ***H***, b the least value of y in ***H'***.

105. If we pass to polar cds., putting $\rho\cos\theta$, $\rho\sin\theta$ for x, y, there results on reduction and inversion:

$$\text{for } \boldsymbol{E}, \quad \rho^2 = \frac{b^2}{1 - e^2\,\overline{\cos\theta}^2}, \quad \text{where} \quad e^2 = \frac{a^2 - b^2}{a^2};$$

$$\text{for } \boldsymbol{H}, \quad \rho^2 = \frac{-b^2}{1 - e^2\,\overline{\cos\theta}^2}, \quad \text{where} \quad e^2 = \frac{a^2 + b^2}{a^2};$$

$$\text{for } \boldsymbol{H'}, \quad \rho^2 = \frac{b^2}{1 - e^2\,\overline{\cos\theta}^2}, \quad \text{where} \quad e^2 = \frac{a^2 + b^2}{a^2};$$

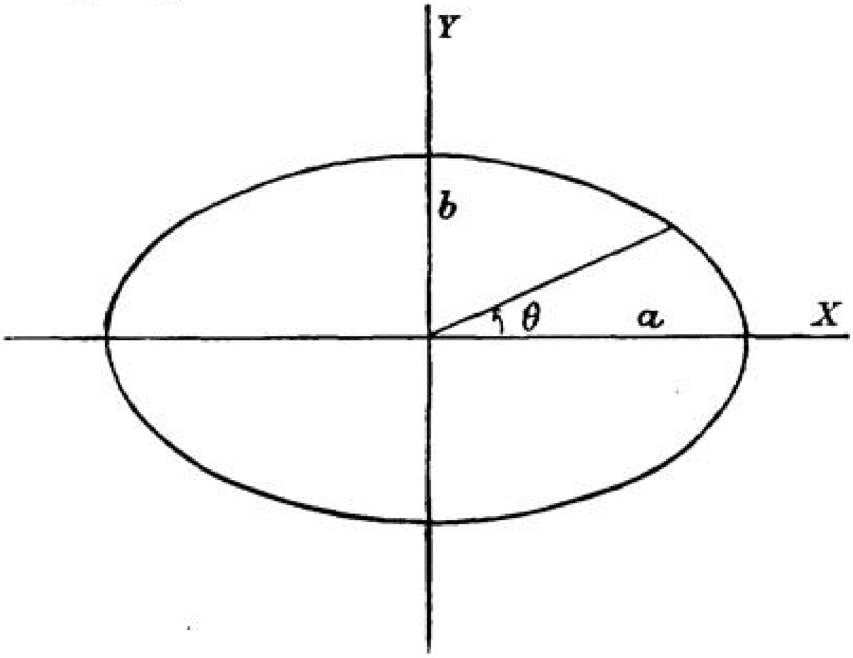
as central polar Eqs. of ***E***, ***H***, ***H'***, one side of the real axis of ***H*** being polar axis. That of ***H*** resp. ***H'*** is got from that of ***E*** by simply changing the sign of b^2 resp. a^2. The geometric meaning of e^2, used here for shortness, will be seen later.

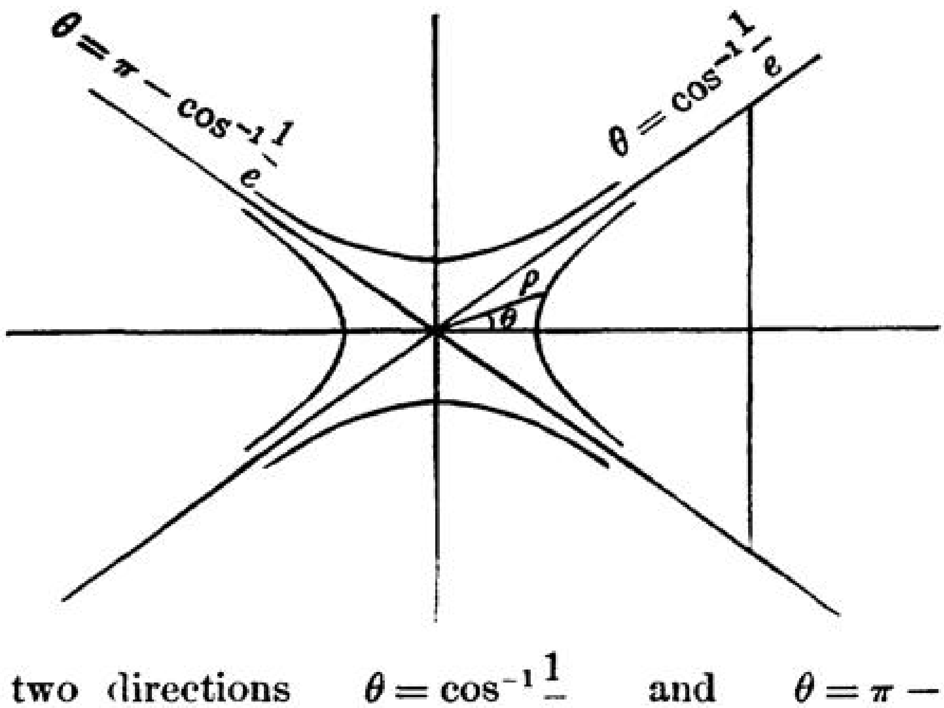
These Eqs. are pure quadratics both in ρ and in $\cos\theta$, also $\overline{\cos\theta}^2 = \overline{\cos(-\theta)}^2 = \overline{\cos(\pi-\theta)}^2$, and two *counter* ρ's make a diameter; therefore,

Diameters like-sloped to an axis are equal, and *equal diameters are like-sloped to an axis.*

106. Let us trace ***E***. For $\theta = 0$, $\rho = a$; as θ increases to $\frac{\pi}{2}$, ρ decreases to b; as θ increases to π, ρ increases to a; as θ increases to $\frac{3\pi}{2}$, ρ decreases to b; as θ increases to 2π, ρ increases to a. The greatest resp. least diameter is $2a$ resp. $2b$.

In ***H***, for $\theta = 0$, $\rho = a$; as θ increases to $\cos^{-1}\frac{1}{e}$, ρ increases to ∞, all values of ρ in ***H'*** being meanwhile imaginary; as θ increases from $\cos^{-1}\frac{1}{e}$ to $\frac{\pi}{2}$ and thence to $\pi - \cos^{-1}\frac{1}{e}$, ρ in ***H'*** decreases from ∞ to b, and thence increases to ∞, all values of ρ in ***H*** being meanwhile imaginary; as θ increases from $\pi - \cos^{-1}\frac{1}{e}$ to π and thence to $\pi + \cos^{-1}\frac{1}{e}$, ρ in H decreases from ∞ to a, and thence increases to ∞, all values of ρ in H' being meantime imaginary; as θ increases from $\pi + \cos^{-1}\frac{1}{e}$ to $\frac{3\pi}{2}$ and thence to $2\pi - \cos^{-1}\frac{1}{e}$, ρ in H' decreases from ∞ to b, and thence increases to ∞, all values of ρ in H being meantime imaginary; as θ increases from $2\pi - \cos^{-1}\frac{1}{e}$ to 2π, ρ in H decreases from ∞ to a, all values of ρ in H' being meantime imaginary.





The two directions $\theta = \cos^{-1}\frac{1}{e}$ and $\theta = \pi - \cos^{-1}\frac{1}{e}$ with their counters correspond to

$$\tan\theta = +\frac{b}{a}, \quad \text{and} \quad \tan\theta = -\frac{b}{a},$$

which are the direction-coefficients of the *asymptotes:*

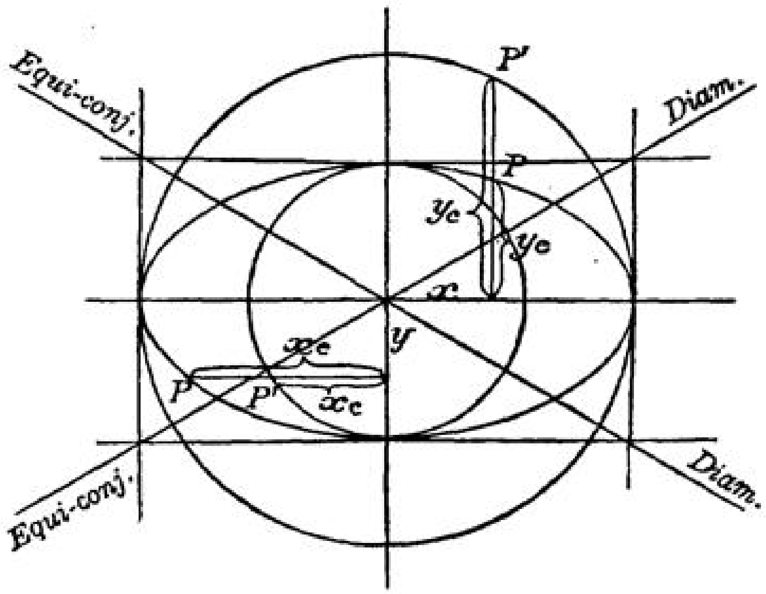
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

Hence the two ***H***'s have common asymptotes, and along these asymptotes they close in upon each other at ∞.

107. Solved as to y resp. x the Eq. of ***E*** is

$$y_e = \frac{b}{a}\sqrt{a^2 - x^2} \quad \text{resp.} \quad x_e = \frac{a}{b}\sqrt{b^2 - y^2}.$$

Now $y_c = \sqrt{a^2 - x^2}$ *resp.* $x_c = \sqrt{b^2 - y^2}$ is the Eq. of a circle about the centre (origin), radius a resp. b, which may be called the *major* resp. *minor circle* of the ***E***. For any value of x the corresponding values of y in the ***E*** and the major circle are in the ratio $y_e : y_c = b : a$. Hence the ***E*** is the *orthogonal projection of its major circle under the* $\measuredangle \cos^{-1}\frac{b}{a}$.



So the minor circle is a like projection of the ***E***. Think the surfaces of ***E*** and the major circle made up of elementary trapezoids, or covered with threads $\perp$ to the common diameter, corresponding elements of the two surfaces will have the fixed ratio $b : a$; hence the whole areas will have that ratio; i.e.,

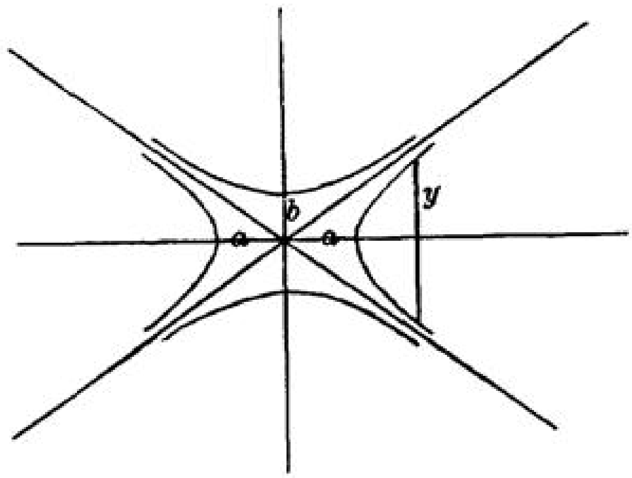
$$\text{area of } \boldsymbol{E} = \frac{b}{a} \cdot \pi a^2 = \pi ab = \sqrt{\pi a^2 \cdot \pi b^2}$$

equals the geometric mean of the areas of major and minor circles.

The student can easily convince himself that the ratio of the projection of any plane area to the area projected is the cosine of the angle of projection.

The Eqs. of $\boldsymbol{E}$ and $\boldsymbol{H}$ solved as to y: $y = \frac{b}{a}\sqrt{a^2 - x^2}$ and $y = \frac{b}{a}\sqrt{x^2 - a^2}$ declare that any ordinate is the $\frac{b}{a}$th part of the geometric mean of the segments into which it cuts the major axis; for the segments are $a + x$, $a - x$ in the $\boldsymbol{E}$, and $x + a$, $x - a$ in the $\boldsymbol{H}$. In the $\boldsymbol{E}$ the section is *inner;* in the $\boldsymbol{H}$ it is *outer.*

For $a = b$, the $\boldsymbol{E}$ reduces to a circle or equiaxial $\boldsymbol{E}$. The *equiaxial* $\boldsymbol{H}$ is $x^2 - y^2 = a^2$. It corresponds to the circle, and is called also *equilateral* or *rectangular*, since its asymptotes are $\perp$. It is congruent with its $\boldsymbol{H}'$: $x^2 - y^2 = -a^2$, and falls on it when turned through $90°$. As in the circle, so in the equiaxial $\boldsymbol{H}$, *any two conjugate diameters are equal.*



108. By Art. 93 the direction-coefficients of two conjugate diameters are connected by the relation, $s' = -\dfrac{k + hs}{h + js}$; or, if *conjugate diameters* be axes, and so $h=0$, $ss'=-\frac{k}{j}$; i.e., *their product equals the negative ratio of the coefficients of* x^2 and y^2; hence, in the present form of the Eqs. of ***E*** *resp.* ***H***,

$$ss'=-\frac{b'^2}{a'^2} \text{ resp. } ss'=\frac{b'^2}{a'^2},$$

or $$\tan\theta\cdot\tan\theta'=-\frac{b^2}{a^2} \text{ resp. } =+\frac{b^2}{a^2}.$$

Hence $\tan\theta$ and $\tan\theta'$ are unlike-signed in ***E***, like-signed in ***H***; i.e., θ and θ' lie in *adjacent* quadrants in ***E***, in the *same* quadrant in ***H***; i.e., of two conjugate diameters of an ***E***, one lies in first and third, one in second and fourth, quadrants; but of an ***H***, both lie in first and third or both in second and fourth.

In ***E***, if $\tan\theta=\pm\frac{b}{a}$, $\tan\theta'=\mp\frac{b}{a}$; i.e., when one of two conjugate diameters of an ***E*** is one diagonal of the rectangle of the tangents ∥ to the axes, the other is the other diagonal. This pair of diameters of the ***E*** are named *equi-conjugate*.

In ***H***, if $\tan\theta=\pm\frac{b}{a}$, $\tan\theta'=\pm\frac{b}{a}$; i.e., *two conjugate diameters fall together* on each diagonal of the rectangle of the tangent ∥ to the axis; i.e., *on each of the asymptotes;* hence *each asymptote is a self-conjugate diameter.*

In ***E***, as θ increases, θ' increases; in ***H***, as θ increases, θ' decreases.

109. *The Eq. of the tangent* to ***E*** or ***H*** is (the *upper* sign going with ***E***)

$$\frac{x_1x}{a'^2}\pm\frac{y_1y}{b'^2}=1, \quad \text{or} \quad \frac{x_1x}{a^2}\pm\frac{y_1y}{b^2}=1.$$

The Eq. of the diameter through (x,y) is $\frac{x}{y}=\frac{x_1}{y_1}$. Its conjugate is ∥ to the tangent, and goes through the centre; hence its Eq. is

$$\frac{x_1x}{a^2}\pm\frac{y_1y}{b^2}=0.$$

To find the Cds. x_2, y_2 of an end of this conjugate, combine its Eq. with the Eq. of the curve, thus:

$$\frac{y^2 y_1^2 a^2}{x_1^2 b^4} \pm \frac{y^2}{b^2} = 1, \quad \text{or} \quad \frac{a^2 y^2}{b^2 x_1^2} \left\{ \frac{y_1^2}{b^2} \pm \frac{x_1^2}{a^2} \right\} = 1.$$

For ***E*** resp. ***H*** the parenthesis { } is +1 resp. −1; hence, for ***E***,

$$x_2 = \mp \frac{a}{b} y_1, \quad y_2 = \pm \frac{b}{a} x_1;$$

for ***H***,

$$x_2 = \pm i \frac{a}{b} y_1, \quad y_2 = \mp i \frac{b}{a} x_1.$$

Again we see only one of two conjugate diameters has real ends on an ***H***.

110. Plainly, if the x (or y) of one end of a diameter is known, the diameter itself is known as one of two equal diameters like-sloped to the X-axis; hence we can *express any squared half-diameter through the x of its end.* If (x_1, y_1) be the end of a' in an ***E***, then

$$a'^2 = x_1^2 + y_1^2 = x_1^2 + \frac{b^2}{a^2}(a^2 - x_1^2) = b^2 + \frac{a^2 - b^2}{a^2} x_1^2 = b^2 + e^2 x_1^2.$$

By Art. 102 $\quad a'^2 + b'^2 = a^2 + b^2, \quad$ hence $\quad b'^2 = a^2 - e^2 x^2.$

Now $\quad b'^2 = x_2^2 + y_2^2; \quad$ hence, by Art. 109,

$$\frac{a^2}{b^2} y_1^2 + \frac{b^2}{a^2} x_1^2 = a^2 - e^2 x^2.$$

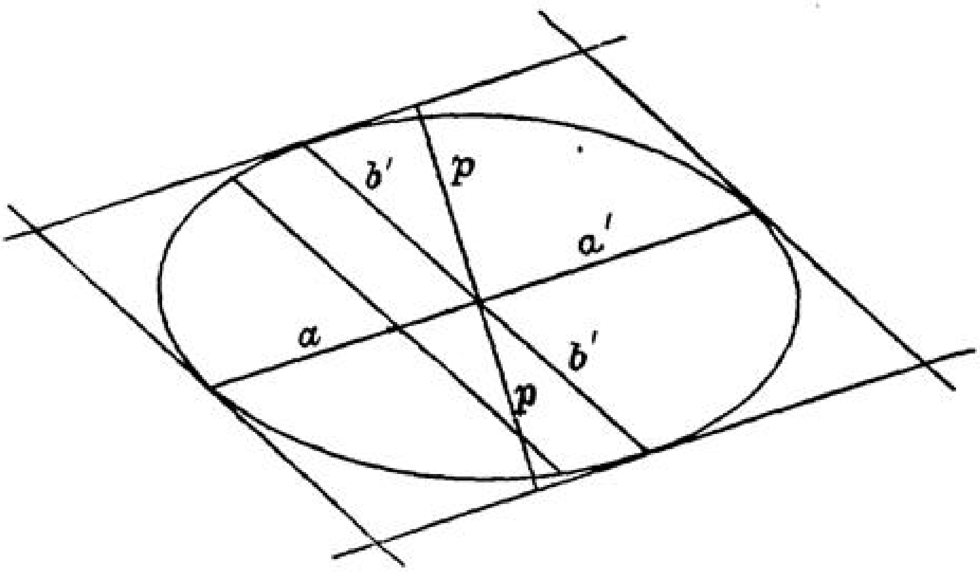
Changing the signs of b^2 and b'^2, we get for the ***H***,

$$a'^2 = -b^2 + e^2 x_1^2, \quad -b'^2 = a^2 - e^2 x_1^2 = -\left(\frac{a^2}{b^2} y_1^2 + \frac{b^2}{a^2} x_1^2\right).$$

111. By Art. 102 the area of the parallelogram of two conjugate half-diameters, and therefore its fourfold: the area of the parallelogram of the tangents through the ends of two conjugate diameters, is constant. If ϕ be one ∡ between these diameters. then this area is $4\,a'b'\sin\phi$ and $=4\,ab$; whence

$$\sin\phi=\frac{ab}{a'b'}.$$

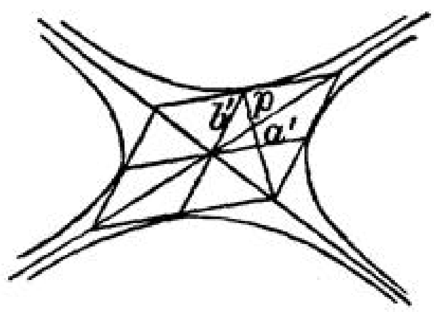
Hence ϕ is *least* when $a'b'$ is *greatest;* and $a'b'$ is *greatest* when $a^2+b^2+2a'b'$, which $=a'^2+b'^2-2\,a'b'$, which $=(a'-b')^2$,



is *least;* i.e., when $a'=b'$; i.e., when *the diameters are the equi-conjugates.* For equi-conjugates,

$$\sin\phi=\frac{ab}{a'b'}=\frac{2\,ab}{a^2+b^2}.$$

Of course this last reasoning is necessary and applicable only in case of the ***E***; in the ***H*** the ∡ between conjugates is least when they fall together in an asymptote.



If $2\,a'$, $2\,b'$ be two conjugate diameters, $2\,p$ the distance between the tangents ∥ to $2\,a'$, then the constant area is

$$2a'\cdot 2p=4\,ab,\qquad\text{or}\qquad p=\frac{ab}{a'};$$

i.e., *the distance from the centre to a tangent is a fourth proportional to the half-diameter ∥ to the tangent and the half-axes.*

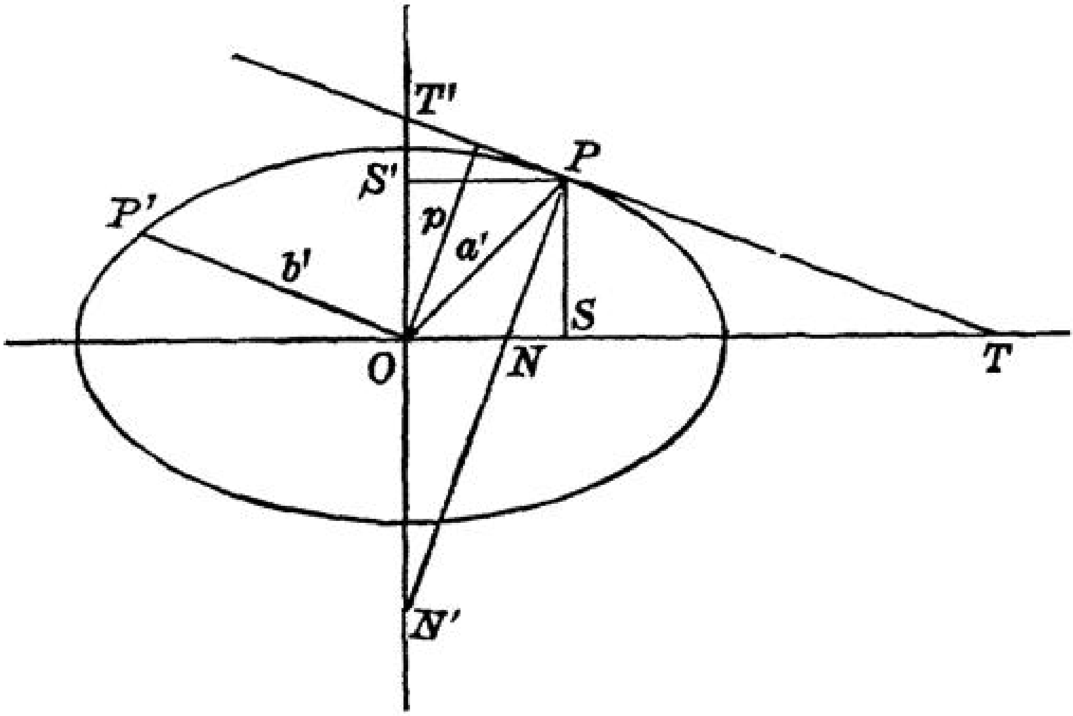
112. The definition of the normal (Art. 73) yields as its Eq.

$$\frac{a^2}{x_1}(x-x_1)\mp\frac{b^2}{y_1}(y-y_1)=0;\quad \text{or,}\quad \frac{a^2}{x_1}\cdot x\mp\frac{b^2}{y_1}\cdot y=a^2\mp b^2.$$

The intercepts of the tangent on the axes are $\frac{a^2}{x_1}$ and $\pm\frac{b^2}{y_1}$; those of the normal are

$$\frac{a^2\mp b^2}{a^2}x_1 \quad\text{and}\quad \frac{a^2\mp b^2}{\mp b^2}y_1.$$

The product of corresponding intercepts of T and N is a constant: $\pm(a^2\mp b^2)$.



The intercepts *on the tangent* between the point of touch and the X- resp. Y-axis may be named X- resp. Y-*tangent* (*lengths*).

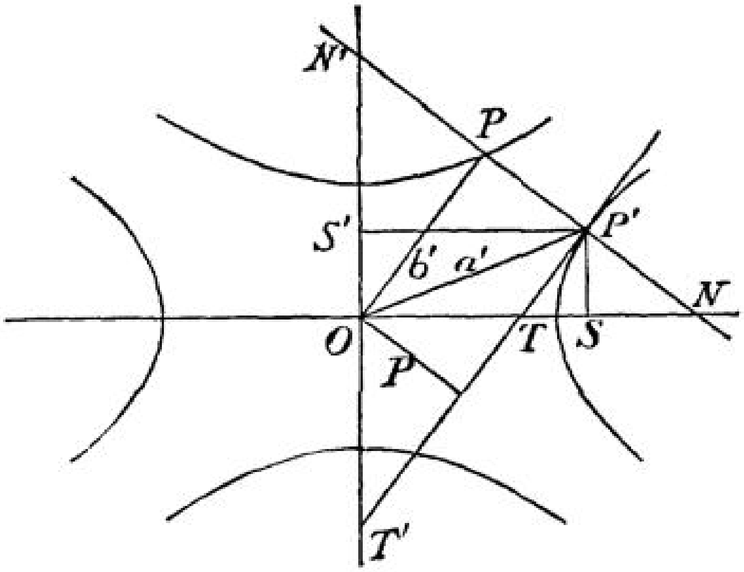
To like intercepts on the normal like names are given.

The projections of these tangent and normal-lengths, each on its own axis, are named sub-tangents and sub-normals.

The following table needs no explanation:

$$X\text{-subtangent}=\frac{a^2}{x_1}-x_1=\frac{a^2-x_1^2}{x_1}=\frac{a^2y_1}{\pm b^2x_1}\cdot y_1;\qquad (1)$$

$$Y\text{-subtangent}=\frac{b^2}{y_1}-y_1=\frac{b^2-y_1^2}{y_1}=\frac{\pm b^2x_1}{a^2y_1}\cdot x_1;\qquad (2)$$



$$X\text{-subnormal} = x_1 - \frac{a^2 \mp b^2}{a^2} x_1 = \pm \frac{b^2}{a^2} x_1; \qquad (3)$$

$$Y\text{-subnormal} = y_1 - \frac{a^2 \mp b^2}{\mp b^2} y_1 = \pm \frac{a^2}{b^2} y_1; \qquad (4)$$

$$X\text{-tangent} = \left\{ y_1^2 + \frac{a^4 y_1^2}{b^4 x_1^2} y_1^2 \right\}^{\frac{1}{2}} = \frac{a y_1}{b x_1} \left\{ \frac{b^2}{a^2} x_1^2 + \frac{a^2}{b^2} y_1^2 \right\}^{\frac{1}{2}}$$

$$= \frac{a y_1}{b x_1} \cdot b'; \qquad (5)$$

$$Y\text{-tangent} = \left\{ x_1^2 + \frac{b^4 x_1^2}{a^4 y_1^2} x_1^2 \right\}^{\frac{1}{2}} = \frac{b x_1}{a y_1} \left\{ \frac{a^2}{b^2} y_1^2 + \frac{b^2}{a^2} x_1^2 \right\}^{\frac{1}{2}}$$

$$= \frac{b x_1}{a y_1} \cdot b'; \qquad (6)$$

$$X\text{-normal} = \left\{ y_1^2 + \frac{b^4}{a^4} x_1^2 \right\}^{\frac{1}{2}} = \frac{b}{a} \left\{ \frac{a^2}{b^2} y_1^2 + \frac{b^2}{a^2} x_1^2 \right\}^{\frac{1}{2}}$$

$$= \frac{b}{a} \cdot b'; \qquad (7)$$

$$Y\text{-normal} = \left\{ x_1^2 + \frac{a^4}{b^4} y_1^2 \right\}^{\frac{1}{2}} = \frac{a}{b} \left\{ \frac{b^2}{a^2} x_1^2 + \frac{a^2}{b^2} y_1^2 \right\}^{\frac{1}{2}}$$

$$= \frac{a}{b} \cdot b'. \qquad (8)$$

Hence these results are evident:

(1) Product of ST's = product of SN's = product of Cds. of point of touch.

(2) Product of T's = product of N's = squared half-diameter ∥ to tangent.

(3) Product of X- resp. Y-normal by central distance of tangent = b^2 resp. a^2.

113. As the tangent is but a special case of the polar, so the normal may be subsumed under the more general concept of a ⊥ *through the pole to the polar.* In lieu of a better, give this ⊥ the name **Perpolar**. Since the Eq. of the polar has the same form as that of the tangent, the Eq. of the perpolar has the same form as that of the normal. As the pole glides along a RL., the polar turns about (envelopes, enwraps) a point; but in the Eq. of the perpolar, which may be written

$$a^2y_1x \mp b^2x_1y = (a^2 \mp b^2)x_1y_1,$$

the parameters x_1, y_1 do *not* appear linearly; hence, when connected by some linear relation, it will not in general be possible to eliminate one from the Eq. of the perpolar and leave the other in first degree only; i.e., as the pole (x_1, y_1) glides along a RL., the perpolar will not in general turn about a point, but about some curve. But in *three* cases it is possible: when x_1 is constant, when y_1 is constant, when $y_1 : x_1$ is constant; i.e., *when the pole moves on a RL. ∥ to either axis or through the centre, the perpolar turns about a point,* the **perpole** of the RL.

If the RL. be ∥ to the Y-axis, x_1 is constant, and the pole of the RL is on the X-axis, distant $a^2 : x_1$ from the centre; then the perpole is also on the X-axis, distant e^2x_1 from the centre. (This is seen at once on writing the Eq. of the perpolar thus:

$\mp b^2x_1y + y_1(a^2x - \overline{a^2 \mp b^2x_1})$. For x_1 constant, this is the Eq. of a pencil of RLs. whose base-lines are $y = 0$, i.e., the X-axis, and $a^2x - (a^2 \mp b^2)x_1 = 0$, i.e., $x = \frac{a^2 \mp b^2}{a^2}x_1 = e^2x_1$; ∴ the perpole is $[e^2x_1, 0]$). The product of these two central distances is the constant a^2e^2; hence, *poles and perpoles of RLs. ∥ to the Y-axis form an involution of points* on the X-axis, whose *centre* is the *centre* of the conic, whose *foci* are distant $\boldsymbol{ae}$ from the centre.

Likewise it is proved that poles and perpoles of RLs. ∥ to the X-axis are in involution on the Y-axis, but the constant product of distances of a pair from the centre is $-a^2e^2$. Hence if either pair of foci are real, the other are imaginary.

The student will readily see that poles and perpoles of all RLs. through the centre lie on the RL. at ∞.

114. The foci of the involutions on the axes are called **foci** *of the curve; hence a centric conic has* **four** *foci:* two *real*, two *imaginary*. They enjoy important properties.

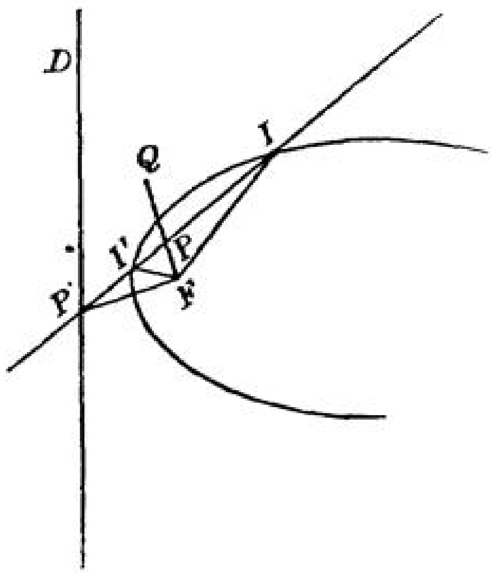
The central distance, ae, of a real focus is called the **linear eccentricity** of the conic; e itself is the **eccentricity** proper. It is the ratio of the central distance of a focus to the half-axis on which the focus lies. Now in case of the imaginary focus the central distance of the focus is imaginary in both ***E*** and ***H***; but the half-axis is real in ***E*** and imaginary in ***H***; hence their quotient is imaginary in ***E*** but real in ***H***; i.e., *one eccentricity is real, one imaginary* in ***E***, *both are real* in ***H***.

The *polar* of a *focus* is called a **Directrix**. Suppose the real foci $(\boldsymbol{ae}, \mathbf{0})$, $(-\boldsymbol{ae}, \mathbf{0})$ on the X-axis; the *directrices* are

$$x = \pm \frac{a}{e}.$$

By Art. 113 a focus is a double point in which have fallen together pole and perpole of a certain RL., the directrix. As a pole glides along this directrix, both its polar and its perpolar turn about the focus *always ⊥ to each other*.

Call the tract from a focus to a point a *focal radius* of that point, and any RL. through a focus a *focal chord*. From any point P' of the directrix draw a chord cutting the conic at I and I'; to the pole Q of this chord, and from the focus F, draw FQ cutting the chord at P. By Art. 71, since the polar of P' is FQ, P', I', P, I form an harmonic range; hence $F\{P'I'PI\}$ is an harmonic pencil. But FP' is clearly the perpolar of P', as this perpolar must go through P' and through F. Hence FP and FP' are ⊥; hence they halve the ∡s of FI and FI' (Art. 41). Hence, *the focal radius of the pole of a chord halves the ∡ at the focus subtended by the chord.*



115. By Art. 113 polar and perpolar cut the axis of involution in a pair of conjugate points, harmonic with the foci; hence the focal radii of the intersection of polar and perpolar form with these two an harmonic pencil; and since these two are ⊥, *they halve the angles between the focal radii.*

When polar and perpolar are tangent and normal, their intersection is the pole, the point of tangence on the conic; hence *tangent and normal halve the ∡s of the focal radii of the point of touch.*

The normal halves the inner resp. outer ∡ in the ***E*** resp. ***H***; hence *an* ***E*** *and an* ***H*** with the same foci, i.e., *confocal*, *are ⊥ to each other.*

116. These relations of position imply several relations of size: the intercept between the foci being $2\,ae$, the X-intercept of the normal being e^2x (Art. 112), the segments of the focal intercept are $ae + e^2x_1$, $ae - e^2x_1$ in ***E***, where the normal cuts it innerly, and $e^2x_1 + ae$, $e^2x_1 - ae$ in ***H***, where the normal cuts it outerly.

Hence, as the focal radii r, r' are proportional to these segments,

$$\frac{r}{r'} = \frac{a + ex_1}{a - ex_1} \text{ in the } \boldsymbol{E}\text{, and } \frac{r}{r'} = \frac{ex_1 + a}{ex_1 - a} \text{ in the } \boldsymbol{H};$$

or $$\frac{r + r'}{r - r'} = \frac{2a}{2ex_1} \text{ in the } \boldsymbol{E}\text{, and } \frac{r + r'}{r - r'} = \frac{2ex_1}{2a} \text{ in the } \boldsymbol{H}.$$

But, plainly,

$$r^2 - r'^2 = (ae + x_1)^2 - (ae - x_1)^2 = 4aex_1,$$

or $$(r + r')(r - r') = 2a \cdot 2ex_1;$$

hence $r + r' = 2a, \quad r - r' = 2ex_1, \quad r = a + ex_1, \quad r' = a - ex_1,$

$$rr' = a^2 - e^2x_1^2 = b'^2;$$

resp. $r + r' = 2ex_1, \quad r - r' = 2a, \quad r = ex_1 + a, \quad r' = ex_1 - a,$

$$rr' = e^2x_1^2 - a^2 = -b'^2.$$

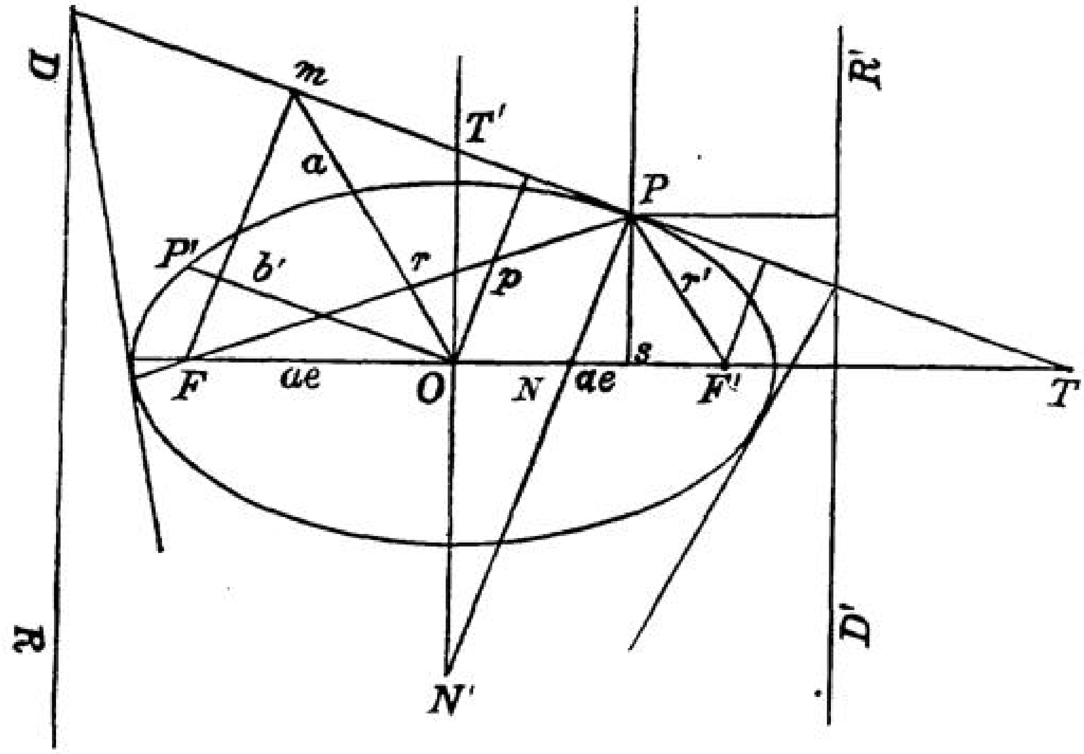
Or, *in the* ***E*** resp. ***H***, *the sum* resp. *difference of the focal radii of a point is a constant*, namely, the *major* resp. *real axis;* and *in both* ***E*** and ***H*** *the product of the focal radii of a point is the squared half-diameter conjugate to the diameter through the point.*

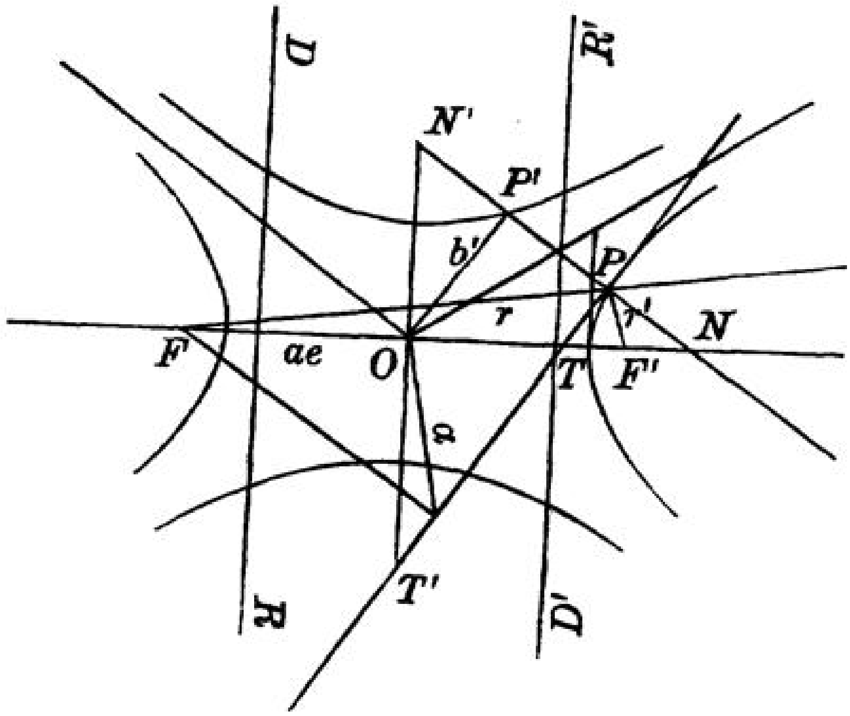
The distance of (x_1, y_1) on the ***E*** from the directrix $x = \frac{a}{e}$ is clearly $\frac{a}{e} - x_1$, and the distance of the same point from the focus is $a - ex_1$; i.e., *in the* ***E*** *the ratio of the distances of a point from focus and directrix is a constant*, the eccentricity. Plainly the like holds for the ***H***. In ***E*** this ratio is < 1, in ***H*** it is > 1.

Let the student show that the locus of a point the sum resp. difference of whose distances from two fixed points is constant is an ***E*** resp. ***H***; also, the locus of a point the ratio of whose distances from a fixed point and a fixed RL. is a constant < 1 resp. > 1 is an ***E*** resp. ***H***.

The sum resp. difference of the focal ⊥s on a tangent is twice the central ⊥ on the tangent; i.e., it is $2a \cdot \frac{b}{b'}$; the sum resp. difference of the focal radii is $2a$; also, the ratio of focal ⊥

to focal radius is the same for the two foci, being the sine of the slope of radius to tangent; hence this ratio, this sine is $\frac{b}{b'}$.





Dividing the central ⊥ on the tangent by this ratio, we get the central distance to a tangent, measured ‖ to a focal radius to the point of touch, namely, a. Hence *the locus of the section* *of a tangent and a diameter* ∥ *to a focal radius to the point of touch* is *the major circle.* The same major circle is the locus of the foot of the focal ⊥ on the tangent; for the △s $PF'N$, MOF are similar, since $\frac{OF}{OM}=e$ and $\frac{F'N}{F'P}=\frac{ae-e^2x_1}{a-ex_1}=e$.

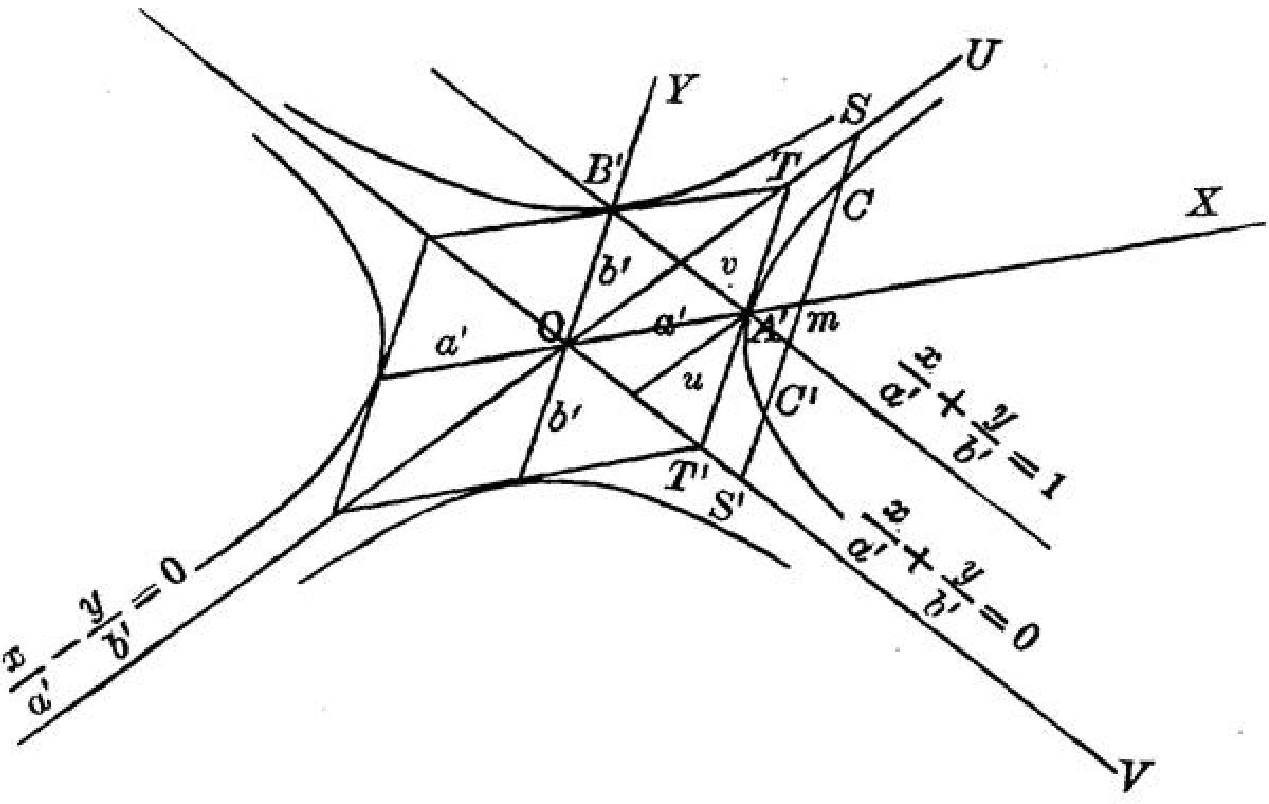
Asymptotic Properties.

117. Thus far the properties of ***E*** and ***H*** have corresponded; but the *asymptotic* properties of the ***H*** have no *real* correspondents in the ***E***, as the asymptotes of the ***E*** are imaginary. Accordingly, in what follows, reference is to the ***H*** alone.

If $\frac{x^2}{a'^2}-\frac{y^2}{b'^2}=1$ be the ***H***, $\frac{x^2}{a'^2}-\frac{y^2}{b'^2}=0$ are its asymptotes.

One of these, $\frac{x}{a'}-\frac{y}{b'}=0$, or $\frac{x}{y}=\frac{a'}{b'}$, is clearly the central diagonal of the parallelogram of the conjugate half-diameters a', b'; the other is the central ∥ to the other diagonal, $\frac{x}{a'}+\frac{y}{b'}=1$.

Hence, given a *pair of conjugate diameters*, we can find the *asymptotes*; or, given the *asymptotes*, we can find the conjugate to any given diameter.



By Art. 42 the RLs. $x=0,\quad y=0;\quad y=\frac{b'}{a'}x,\quad y=-\frac{b'}{a'}x,$ i.e., the *asymptotes and any pair of conjugate diameters form an harmonic pencil.** Hence *the asymptotic intercept of a ∥ to any diameter is halved by its conjugate diameter*, and the intercept between two conjugate diameters of a ∥ to one asymptote is halved by the other. As a special case, the *asymptotic intercept of a tangent is halved at the point of tangence.*

Since the same conjugate diameter that halves the intercept between the asymptotes also halves the chord of the curve, clearly *the intercepts between the asymptotes and curve are* $=$, or, $CS=C'S'$.

From the Eqs. of the curve and the asymptotes there follows:

$$y_a=mS=mS'=\frac{b'}{a'}x,$$

$$y_h=mC=mC'=\frac{b'}{a'}\sqrt{x^2-a'^2};$$

$$\therefore y_a-y_h=CS=C'S'=b'\left\{\frac{x}{a'}-\sqrt{\frac{x^2}{a'^2}-1}\right\},$$

$$y_a+y_h=CS'=C'S=b'\left\{\frac{x}{a'}+\sqrt{\frac{x^2}{a'^2}-1}\right\};$$

$$\therefore CS\cdot CS'=C'S'\cdot C'S=b'^2;$$

i.e., *the product of the distances of any point of an* ***H*** *to the asymptotes*, in any direction, *equals the squared ∥ half-diameters of the* ***H***. Clearly CS can be made large, and so CS' small, at will.

118. The tangent-intercept between the asymptotes TT'' being halved at the point of touch A', the $\triangle\ TOT''=2\ TOA'=$ the parallelogram $A'OB'T$ of the conjugate half-diameters, a'

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* Hence *conjugate diameters form an* **Involution** *of which the asymptotes are the double or focal rays.* Like may be easily proved of the conjugate diameters and imaginary asymptotes of the ***E*** by noting Art. 107 and forming the Determinant of Art. 47.

and b', i.e., = the constant ab. From A' draw to each asymptote a ‖ to the other, and call them u and v; they are the Cds. of A', the asymptotes being axes. With the asymptotes they form a parallelogram which is clearly half the $\triangle\ TOT'$; hence, if ϕ be the ∡ of the asymptotes, we have $uv \sin\phi = ab : 2$. But by Art. 111, since the asymptotes of ***H*** fall on the equi-conjugate diameters of ***E***,

$$\sin\phi = \frac{2ab}{a^2+b^2}; \qquad \therefore uv = \frac{a^2+b^2}{4}.$$

This, the *Eq. of the* ***H*** *referred to its asymptotes*, says the parallelogram of asymptotic Cds. of a point is of constant area.

Focal ⊥s on the asymptotes are clearly equal; the asymptotes being tangents, their product is $-b^2$ (Art. 112); hence each is b in length, but they are counter-directed. This is also plain at once from trigonometric considerations.

Polar Equation of Centric Conic.

119. Take the right focus as pole, the right X-direction as polar axis; then, by Art. 116, $\rho = a - ex$, resp. $\rho = ex - a$ in ***E*** resp. ***H***; x being here reckoned from the centre,

$$x = ae + \rho\cos\theta;$$

hence
$$\rho = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{b^2}{a} : \overline{1+e\cos\theta}$$

$$\text{resp. } \rho = \frac{a(e^2-1)}{1-e\cos\theta} = \frac{b^2}{a} : \overline{1-e\cos\theta}.$$

The expressions for the right focal radius being unlike in ***E*** and ***H***, we might have expected these Eqs. of ***E*** and ***H*** to turn out unlike. This makes the Eqs. just found unhandy. But the *one* expression for the left focal radius is $\rho = a + ex$.

The left X-direction being taken as polar axis and θ reckoned *clockwise* being taken as *positive*, we have $x = -(ae + \rho\cos\theta)$;

$$\therefore \rho = \frac{a(1-e^2)}{1+e\cos\theta} = \pm\frac{b^2}{a} : \overline{1+e\cos\theta} \quad \text{in } \boldsymbol{E} \text{ resp. } \boldsymbol{H}.$$

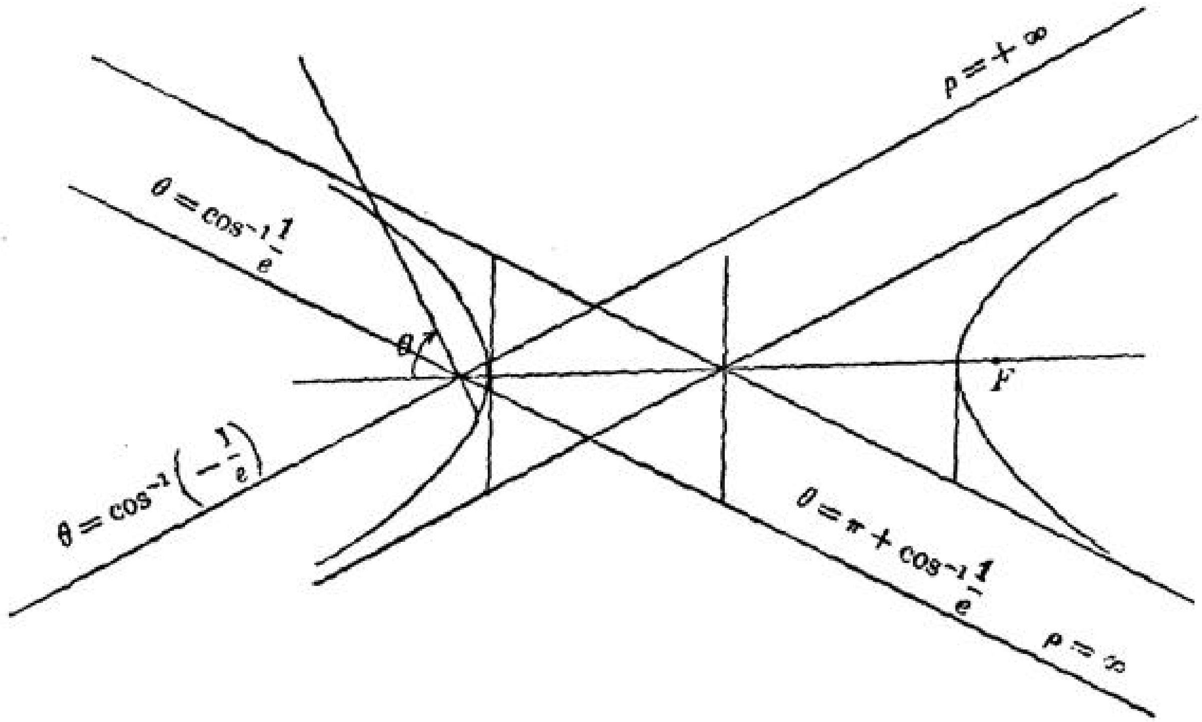
These Eqs. of ***E*** and ***H***, like the central Eqs., differ only in the sign of b^2. For $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, $\rho = \pm\frac{b^2}{a}$. Hence $\frac{2b^2}{a}$ is the *focal chord* $\perp$ *to the axis;* it is called the *parameter* or *latus rectum*.

It is interesting to trace the curve from the polar Eq.

The corresponding values of ρ are unlike-signed in ***E*** and ***H***; in ***E*** ρ is measured *bounding the* $\measuredangle\,\theta$, in ***H*** it is measured *counter* as long as ρ falls out negative. The value of e is not the same in the two Eqs.; in ***E*** it is <1, hence the divisor $1+e\cos\theta$ remains throughout $+$ and finite; but not so in ***H***.

For $\theta = 0$, $\rho = a - ae$; in ***E*** this is $+$, but in ***H*** it is $-$; hence it is reckoned leftward in ***E***, rightward in ***H***. As θ increases to $\cos^{-1}\left(-\frac{1}{e}\right)$, ρ traces out (by its end) the *left lower* branch of ***H***. For $\theta = \cos^{-1}\left(-\frac{1}{e}\right)$, $\rho = -\infty$, and is drawn in the *left lower* asymptotic direction. Just here ρ changes sign, and beginning at $+\infty$, traces out the *right upper* branch of ***H*** till $\theta = \pi$, when ρ sinks to $a + ae$ at the right vertex; thence staying $+$, it rises to $+\infty$, tracing out the *lower right* branch of ***H*** till θ reaches $\pi + \cos^{-1}\frac{1}{e}$. Here again ρ changes sign, becoming $-\infty$, and as θ increases to 2π, it traces out the *left upper* branch of ***H***, reaching the *left* vertex as it reaches 2π. The student himself may readily follow the course in the ***E***.

It is noteworthy that the *right upper* branch of ***H*** is thus seen to be continuous (in ∞) with the *left lower*, and the *right lower* with the *left upper*. We are forced to think the ***H*** thus by this reasoning also: The asymptote touches the ***H*** at ∞ in two counter directions; hence unless it meets each branch in the same two consecutive points at ∞, it must meet ***H*** in four points, which is impossible. While, then, to imagination the ***H*** consists of two distinct branches, to reason it consists of one branch closed in two directions (the asymptotic) at ∞.



Non-Centric Conic: Parabola.

120. By referring ***P*** to a diameter and the tangent through its end as *X*- and *Y*-axes, its Eq. is brought to the simplest form,

$$y^2 = 4\,q'x \qquad \text{(Art. 100).}$$

One and only one diameter is $\perp$ to its conjugates (Art. 94); it is called *principal diameter* or **axis** of ***P***. The Eq. of ***P*** referred to these $\perp$s is called the *vertical* Eq. of ***P***, the origin being the *vertex*, and is written $y^2 = 4\,qx$; $4\,q'$ is called *parameter* of the corresponding diameter, $4\,q$ is *principal parameter*, or simply **parameter**. From these Eqs. we may now draw out all the properties of the ***P***, as is done in most texts; but another method seems directer.

According as $h^2 - kj$ is $< 0,\ = 0,\ > 0$, the conic is ***E***, ***P***, ***H***. Hence, any two of the symbols k, h, j being held fast, as the other changes, the conic becomes in turn an ***E*** of this or that shape, a ***P***, and an ***H*** of this or that shape. ***P*** is thus seen to be a *critical* curve between ***E***'s and ***H***'s, a *border* or *limit* of the two. What kind of a limit, we shall see.

For this investigation of the relation of ***P*** to ***E*** on one hand and ***H*** on the other, the central Eq. of ***E*** and ***H*** is ill suited, as the central Eq. of ***P*** is unmanageable, the centre being at ∞. Since the vertical is the simplest Eq. of ***P***, let us move the origin to (say) the left vertex in ***E*** and ***H***. The Eq. becomes

$$\frac{\overline{x-a}^2}{a^2} \pm \frac{y^2}{b^2} = 1 \quad \text{or} \quad y^2 = \frac{\pm b^2}{a^2}(2ax - x^2).$$

Here $k = \pm \frac{b^2}{a^2}$, $h = 0$, $j = 1$; hence, if either ***E*** or ***H*** is to pass over into ***P***, $\frac{b^2}{a^2}$ must vanish to make $h^2 - kj = 0$; and $\frac{2b^2}{a}$ must stay finite to make the Eq. $y^2 = 4qx$. Now $\frac{2b^2}{a}$ or $4q$ is the focal chord $\perp$ to the axis; hence $\frac{b^2}{a}$ or $2q$ is the ordinate at the focus, y_f, and $\frac{b^2}{2a}$ or q is the focal abscissa x_f, or distance of the focus from the vertex. Accordingly, we keep $\frac{2b^2}{a}$ finite by holding the focus and vertex fixed; to make $\frac{b^2}{a^2}$ vanish, we must let a increase toward ∞; i.e., let the *centre*, and with it the *other focus*, *retire to* ∞. But then

$$\frac{a^2 \mp b^2}{a^2} = 1 \mp \frac{b^2}{a^2} = 1, \quad \text{or} \quad e^2 = 1.$$

Hence we may treat ***P*** as an ***E*** (or ***H***) whose *parameter has kept constant while its centre and one focus have retired to* ∞, or *as an* ***E*** (or ***H***) *whose eccentricity has increased* (or *decreased*) *to* 1. The properties of ***P*** are the properties of ***E*** (or ***H***) *at this limit*, viz.:

121. ***P*** is symmetric as to its axis (Art. 104).

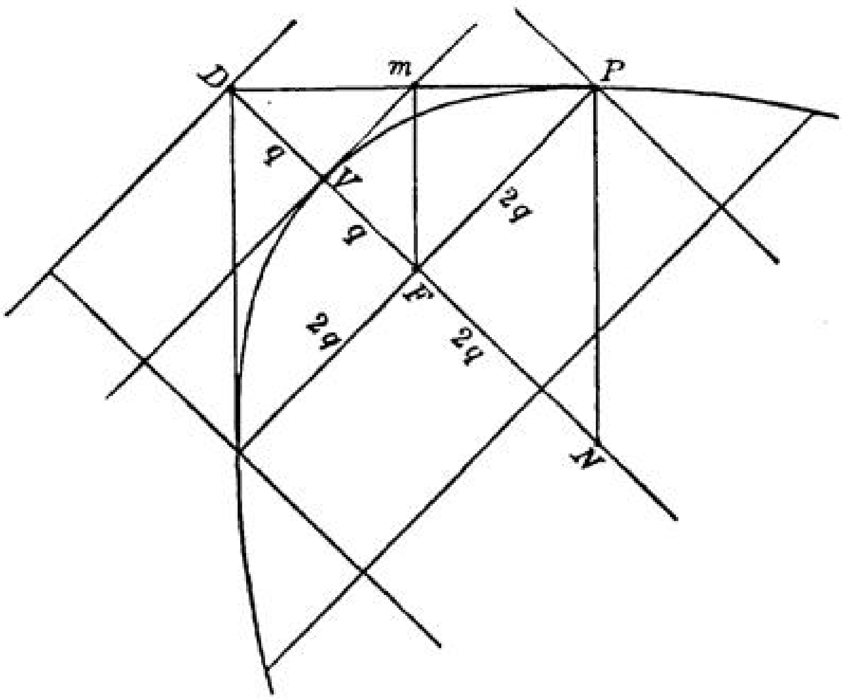
P is the locus of a *point equidistant from focus and directrix.*

The vertex (origin) is distant q from the directrix; hence *any point* (x_1, y_1) of **P** is *distant* $x_1 + q$ *from focus and from directrix.*

The poles of all ∥ chords lie on the diameter of those chords (Art. 98). Also a pole P, the intersection M of its polar and the diameter through it, and the intersections I, I' of the diameter and the curve, form an harmonic range; and as I' is at ∞, I is midway between P and M; i.e., *the intercept on a diameter between a pole and its polar is halved by the curve* ***P***.

If the diameter be the axis, I is the vertex, and the intercept is the subtangent; i.e., *the subtangent is halved at the vertex.*

Hence the subtangent is $2x'$ long; also it is cut by the focus into segments $x_1 + q$ and $x_1 - q$. Hence the *focal distance of the intersection of tangent and X-axis* (axis of **P**) = the *focal distance of the point of tangence* $= x_1 + q$.



Hence the focal ⊥ on the tangent halves the tangent-length; so too does the vertical tangent; hence the *locus of the foot of the focal ⊥ on the tangent is the vertical tangent.*

Further, clearly the focus halves the distance between the intersections of tangent and normal with the axis; hence the whole intercept is $2(x_1+q)$; on taking away the subtangent $2x_1$ there remains the subnormal $=2q$; i.e., *the subnormal in* ***P*** *is the constant half-parameter.*

Plainly a circle about the focus and through the point of touch goes also through the intersections of tangent and normal with the axis.

Polar and perpolar and focal rays through their intersection form an harmonic pencil (Art. 114). The second focus being at ∞, the second focal ray is the diameter through the intersection. Hence *any polar and perpolar halve the* $\measuredangle$*s between the focal ray and the diameter through their intersection.* As a special case, the *tangent and normal halve the* $\measuredangle$*s between the focal ray and the diameter through any point of a* ***P***. Hence, too, since the diameter meets the axis at ∞, *the focus halves the axial intercept between the conjugates:* polar and perpolar, or specially, *tangent and normal.* This has already been proved geometrically in the special case.

122. All these relations are readily drawn out from the Eqs. of tangent and normal:

$$yy_1=2q(x+x_1) \quad \text{and} \quad (y-y_1)2q+y_1(x-x_1)=0,$$

a useful exercise left for the student. Eliminating x_1 by the relation $y_1^2=4qx_1$, we get

$$yy_1=2qx+\frac{y_1^2}{2} \quad \text{and} \quad (y-y_1)2q+y_1\left(x-\frac{y_1^2}{4q}\right)=0.$$

Solved as to y_1, x and y being treated as known, these Eqs. yield *two* resp. *three* roots, values of y, i.e., ordinates of the points where *tangents* resp. *normals* drawn through (x, y) *meet the* ***P***; hence, *through any point* may be drawn *two* tangents and *three* normals to ***P***. The tangents are real and separate, real and coincident, or imaginary, according as

$$y^2-4qx \text{ is } >0,\ =0, \text{ or } <0;$$

i.e., according as the point from which they are drawn be *without*, *upon*, or *within* the ***P***. The sum of the roots is $2y$; i.e., the ordinate of the point through which the tangents are drawn is the half-sum of the ordinates of the points of touch; i.e., the point is on the diameter of the chord of contact, as already known.

The reduced Eq. of the normal is

$$y_1^3 + 4q(2q - x)y_1 - 8q^2y = 0.$$

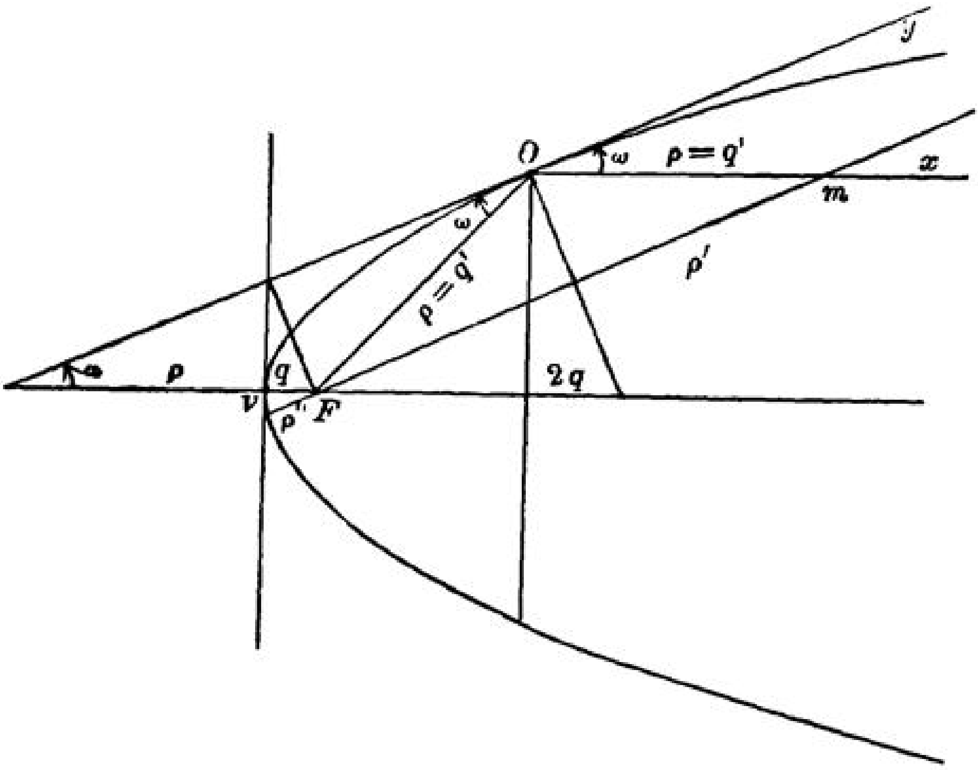
The absence of the term in y_1^2 shows that its coefficient, the sum of the roots, vanishes; or, $y_1' + y_1'' + y_1''' = 0$; i.e., *the sum of the ordinates of the points where three normals to a **P**, through a point, meet the **P** is* 0; or each is the negative sum of the other two. The sum of the ordinates of the ends of ∥ chords is constant, namely, the double ordinate of their diameter; hence all third normals through the intersections of pairs of normals at the ends of ∥ chords cut the ***P*** at points having the same y_1, i.e., at the *same point;* i.e., are the *same normal;* i.e., *pairs of normals at the ends of* ∥ *chords meet on a third normal.* To find this normal, draw one of the ∥ chords through the vertex; then the point symmetric as to the axis with the other end of the chord is the point of ***P*** through which the third normal goes.

One of the normals is always real, wherever (x, y) be taken; the other two are real and separate, real and coincident, or imaginary, according as $y^2 - 4qx < 0$, $= 0$, or > 0; i.e., according as the point they are drawn from is *within*, *upon*, or *without* the ***P***. This is also clear geometrically, since plainly real intersections of normals take place only within the curve.

123. In the vertical Eq. of ***P***, $y^2 = 4qx$, the principal parameter $4q$ is the double focal ordinate, or focal chord. In the Eq., $y^2 = 4q'x$, of ***P*** referred to any diameter and the tangent at its end, on putting $x = q'$, the parameter $4q'$ also appears as a double ordinate; but is it also a focal chord?

Be ρ the focal ray to the origin or point of tangence, O, ω the tangent's slope to the axis, ρ' the focal ray ∥ to the tangent and sloped ω, ρ'' the counter-ray sloped $\overline{\pi + \omega}$; then $\rho = x' + q$, $\rho' = 2\,q : \overline{1 - \cos\omega}$, $\rho'' = 2\,q : \overline{1 + \cos\omega}$, $\rho' + \rho'' = 4\,q : \overline{\sin\omega}^2$.

The second Eq. is got from the polar Eq. of the ***E*** by putting $e = 1$, $\theta = \pi - \omega$; the third from the second by putting $\overline{\pi + \omega}$ for ω. Clearly $\rho' + \rho''$ or $4\,q : \overline{\sin\omega}^2$ is the focal chord $\parallel$ to the tangent; the x of this chord is the focal distance of the intersection of tangent and axis, i.e., the focal distance of the point of



tangence, i.e., ρ. Now project ρ on the focal $\perp$ to the tangent, and project this projection on the axis; by Art. 121 the last projection is q; i.e., $\rho = q : \overline{\sin\omega}^2$; $\therefore\ 4\,\rho = \rho' + \rho'' =$ focal chord. Hence the abscissa to the focal ordinate is half that ordinate; but this is the property of the abscissa q'. Hence $4\,q'$ is the double focal ordinate, or focal chord, and $= 4\,q : \overline{\sin\omega}^2$, where ω is the axial angle or slope of the tangent.

CHAPTER VI.

SPECIAL METHODS AND PROBLEMS.

Magic Equations of Tangents and Normals.

124. Thus far the Eq. of the tangent has been expressed through the Cds. of the point of touch. This Eq. does not in itself determine the tangent, but only by help of an understood Eq. of condition declaring the point of touch to be on the curve; without this latter, it were the more general Eq. of a polar.

Thus, $y_1y = 2q(x_1 + x)$ touches the ***P*** $y^2 = 4\,qx$ only in case (x_1, y_1) be on ***P***, i.e., only in case $y_1^2 = 4\,qx_1$; otherwise, it is but the polar of (x_1, y_1) as to the ***P*** $y^2 = 4\,qx$. This implied Eq. greatly cumbers operations about the tangent.

Where not the point of tangence but only the direction of the tangent is involved, this cumbrance may be avoided by expressing the Eq. of the tangent through the direction-coefficient s as the parameter of the Eq. This form of the Eq. of the tangent is called the *magic equation* of the tangent.

It may be got by putting for y its value $sx + d$ in the general Eq. of the conic, and expressing the condition of equal roots of the quadratic in x, whence may be found d in terms of s. But this is tedious. Better is it to get the special forms for ***E***, ***H***, ***P*** separately. Thus, after the above substitution in $y^2 = 4\,qx$, the roots are equal when $s^2d^2 = (sd - 2\,q)^2$, or when $d = q : s$;

$$\therefore\ y = sx + \frac{q}{s}$$

is the magic Eq. sought; s being thought as changing, it is the Eq. of a family of RLs. touching the ***P*** $y^2 = 4\,qx$.

In finding the magic Eq. for the ***E***, we may exemplify another method. Solved as to y the ordinary Eq. is

$$y = -\frac{b^2}{a^2} \cdot \frac{x_1}{y_1} \cdot x + \frac{b^2}{y_1}.$$

Here $$s = -\frac{b^2}{a^2} \cdot \frac{x_1}{y_1}; \quad d = \frac{b^2}{y_1};$$

square s, multiply by a^2, add b^2; results $d^2 = s^2a^2 + b^2$.

Hence $$y = sx \pm \sqrt{s^2a^2 + b^2}$$

resp. $$y = sx \pm \sqrt{s^2a^2 - b^2}$$

is the magic Eq. for ***E*** resp. ***H***.

N.B. The steps in this elimination are suggested by the reflection that

$$a^2y_1^2 + b^2x_1^2 = a^2b^2.$$

125. Magic Eqs. of normals are easy to find. Thus, in ***P***,

$$s = \frac{2q}{y_1}, \qquad y_1 = \frac{2q}{s}, \qquad x_1 = \frac{q}{s^2}.$$

Substituting in the Eq. of the normal

$$y - y_1 = -\frac{1}{s}(x - x_1),$$

and writing s for $-\frac{1}{s}$, we get as Eq. sought

$$qs^3 + (2q - x)s + y = 0.$$

In ***E***, $$y_1 = b^2 : \sqrt{s^2a^2 + b^2},$$

whence $$x_1 = -s^2a^2 : \sqrt{s^2a^2 + b^2};$$

whence, on substituting as in case of ***P***, there results

$$(y - sx)^2 = s^2(b^2 - a^2)^2 : (s^2b^2 + a^2)$$

as Eq. of normal to ***E***.

Changing b^2 to $-b^2$, we get a like Eq. of normal to ***H***.

These Eqs. are of fourth degree in s; hence may be drawn from any point four normals to an ***E*** or an ***H***.

126. The use of the magic Eq. may be illustrated in finding the locus of the intersection of a pair of ⊥ tangents to an ***E***:

$$y = sx + \sqrt{s^2a^2 + b^2} \quad \text{and} \quad y = -\frac{1}{s}x + \sqrt{\frac{a^2}{s^2} + b^2}.$$

Clear, transpose, square, sum, and divide by $1 + s^2$; results

$$x^2 + y^2 = a^2 + b^2; \quad \text{or, in case of } \boldsymbol{H}, \quad x^2 + y^2 = a^2 - b^2.$$

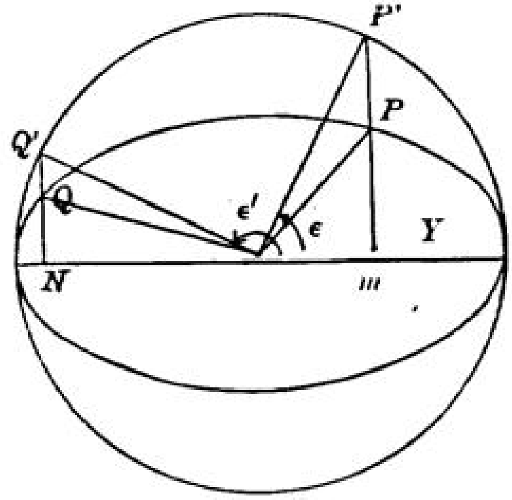
These are named *director-circles* of ***E*** resp. ***H***. If ***E*** and ***H*** be co-axial, the D.C. of each goes through the foci of the other.

Show that the D.C. of ***P*** is the directrix.

The Eccentric Angle.

127. By Art. 107 the ***E*** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the vertical shadow or ∥ projection of the circle $x^2 + y^2 = a^2$. In this projection chords ∥ to the plane of the ***E*** are projected at full length, chords ⊥ to these are shortened in the ratio $b : a$, every other system of ∥ chords are shortened in some ratio between 1 and $b : a$. Clearly the centre of the circle is projected into the centre of the ***E***, hence the diameters of the circle into the diameters of the ***E***. A pair of ⊥ diameters in the circle are conjugate, each halving all chords ∥ to the other; hence their *projections* are *conjugate diameters* of the ***E***, each halving all chords ∥ to the other. Call the diameter ∥ to the plane of the ***E***, which is projected into the axis major of the ***E***, the *axis* of the circle; then the ∡ which any diameter makes with this axis is called the *eccentric angle* of the projection of that diameter. The eccentric ∡ of a point of the ***E*** is the eccentric ∡ of the diameter through it. Hence the *eccentric* ∡s of two *conjugate diameters* differ by 90°.

If the projected circle be turned round its axis through $\cos^{-1}\frac{b}{a}$, it will fall on the major circle of the *E*. If ϵ be the eccentric ∡ of (x, y) on the *E*, then the Cds. (x', y') of the corresponding point of the circle are



$$x' = a\cos\epsilon, \quad y' = a\sin\epsilon;$$

hence $x = a\cos\epsilon, \quad y = b\sin\epsilon$

are the Eqs. of the *E* in terms of ϵ.

128. The eccentric ∡ is especially useful in dealing with chords and tangents. Thus the chord through ϵ_1, ϵ_2 is

$$\begin{vmatrix} x & y & 1 \\ a\cos\epsilon_1 & b\sin\epsilon_1 & 1 \\ a\cos\epsilon_2 & b\sin\epsilon_2 & 1 \end{vmatrix} = 0,$$

which on reduction takes the form

$$\frac{x}{a}\cos\frac{\epsilon_1+\epsilon_2}{2} + \frac{y}{b}\sin\frac{\epsilon_1+\epsilon_2}{2} = \cos\frac{\epsilon_1-\epsilon_2}{2}.$$

Putting $\epsilon_1 = \epsilon_2 = \epsilon$, we get the Eq. of the tangent at ϵ:

$$\frac{x}{a}\cos\epsilon + \frac{y}{b}\sin\epsilon = 1.$$

Replacing x_1 and y_1 in the Eq. of the normal, we get its Eq.

$$\frac{ax}{\cos\epsilon_1} - \frac{by}{\sin\epsilon_1} = a^2 - b^2.$$

The advantage of these Eqs. lies in the fact that the arbitrary ϵ's are free from condition. We may illustrate their use in finding the locus of the intersection of the pair of tangents at the ends of conjugate diameters. Such a pair are

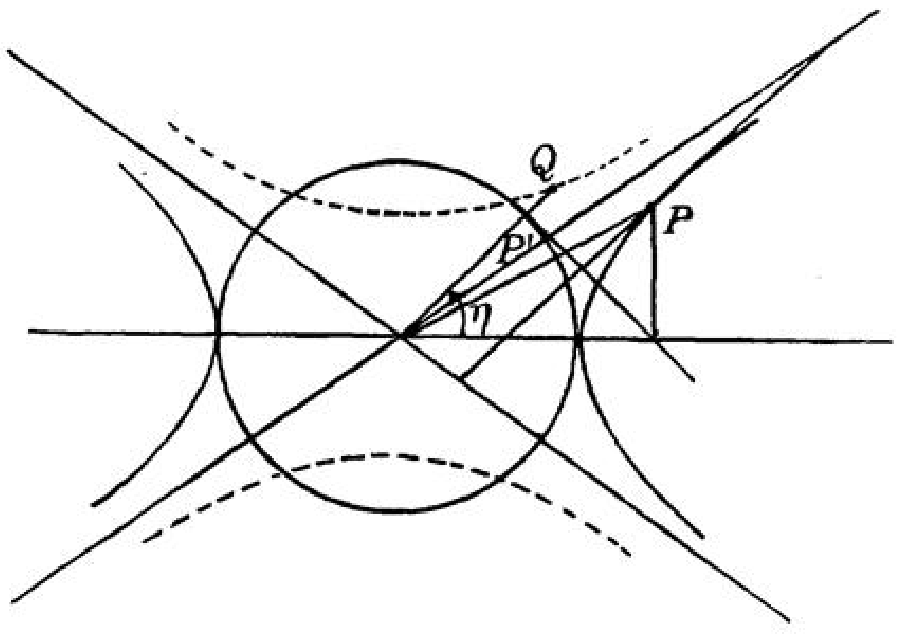
$$\frac{x}{a}\cos\epsilon + \frac{y}{b}\sin\epsilon = 1 \text{ and } -\frac{x}{a}\sin\epsilon + \frac{y}{b}\cos\epsilon = 1.$$

Finding hence $\cos\epsilon$ and $\sin\epsilon$, and placing the sum of their squares equal to 1, we get, on reducing,

$$\frac{x^2}{2a^2}+\frac{y^2}{2b^2}=1,$$

a co-axial **E** with axes multiplied by $\sqrt{2}$.

129. We have seen that *equiaxial* **H** is to **H** in general as the circle (*equiaxial* **E**) is to **E** in general. Any **H** may be thought as a vertical shadow or parallel projection of an equiaxial **H** under $\cos^{-1}\frac{b}{a}(a>b)$. If $a<b$, we may think the relation reversed: the equiaxial **H** the projection of **H** in general. Now since $\overline{\sec\eta}^2-\overline{\tan\eta}^2=1$, if we put $x=a\sec\eta$, as we may, we must have $y=a\tan\eta$ in equiaxial **H**, or $y=b\tan\eta$ in **H** in general. These, then, are the Eqs. of **H** in terms of η.



To construct η we have but to form a right $\triangle$ with *base* a and *hypotenuse* x; the $\measuredangle$ at the base will be η. This is done by drawing from the end of the abscissa x a tangent to the major circle of the **H**. The tangent-length is the corresponding y in the equiaxial **H** and the $\frac{a}{b}$th part of the y in the **H** in general.

The equiaxial ***H'*** $y^2 - x^2 = 1$, is got by exchanging x and y in the equiaxial ***H*** $x^2 - y^2 = 1$; hence its Eqs. are

$$x = a \tan \eta, \quad y = a \sec \eta.$$

In the general ***H'*** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$, y is changed in the ratio $\frac{b}{a}$ while x is unchanged; hence the Eqs. of it are

$$x = a \tan \eta, \quad y = b \sec \eta.$$

In the equiaxial ***H***'s conjugate diameters are equal and like-sloped, the one to the X-, the other to the Y-axis; hence the points corresponding to like values of η in the two pairs of Eqs. are ends of conjugate diameters; after projection conjugate diameters remain conjugate, each still halving all chords ∥ to the other; hence like values of η yield ends of conjugate diameters in the pairs of general Eqs.

Noting the signs of the trigonometric functions, we see that η ranging from 0 to $\frac{\pi}{2}$ yields all points in the right upper branches of ***H*** and ***H'***; η from $\frac{\pi}{2}$ to π yields all points on the left lower branches of both; η from π to $\frac{3\pi}{2}$ yields all on the left upper branch of ***H*** and the right lower branch of ***H'***; η from $\frac{3\pi}{2}$ to 2π yields all on the right lower branch of ***H*** and the left upper branch of ***H'***. Hence the ends of conjugate diameters, answering to like values of η, will be in the same quadrant for η between 0 and π, but in counter-quadrants for η between π and 2π. Observing this, we find as Eqs. of tangents through ends of conjugate diameters in first and third quadrants

$$\frac{x}{a} \sec \eta - \frac{y}{b} \tan \eta = 1, \quad \frac{x}{a} \tan \eta - \frac{y}{b} \sec \eta = -1;$$

whence, on addition to eliminate η, we get

$$\frac{x}{a} - \frac{y}{b} = 0,$$

i.e., one asymptote, as locus of the intersection of the tangents.

Putting $\pi-\eta$ for η in the second Eq., we get

$$-\frac{x}{a}\tan\eta+\frac{y}{b}\sec\eta=-1$$

as Eq. of the tangent through the other end of the second diameter, whence

$$\frac{x}{a}+\frac{y}{b}=0,$$

i.e., the second asymptote, results as locus of the intersection.

130. There is another noteworthy way of expressing the Cds. of a point on an ***H*** through a third variable. As the student may know, sine and cosine may be defined analytically, without any geometric reference, through *exponentials*, thus:

$$\cos\theta=\tfrac{1}{2}(e^{i\theta}+e^{-i\theta}),\qquad \sin\theta=\frac{1}{2i}(e^{i\theta}-e^{-i\theta}),$$

where $i\cdot i=-1$.

All properties of sine and cosine may be drawn out from these definitions with greater ease and generality than from any other.

If instead of i be written 1, the resulting expressions

$$\tfrac{1}{2}(e^{\theta}+e^{-\theta})\qquad\text{and}\qquad\tfrac{1}{2}(e^{\theta}-e^{-\theta})$$

are named resp. *hyperbolic cosine* and *sine* of θ, and may be written $\overline{hc\theta}$ and $\overline{hs\theta}$. We see at once that $\overline{hc\theta}^2-\overline{hs\theta}^2=1$, and hence we may write in ***H***

$$\frac{x}{a}=\overline{hc\theta},\quad \frac{y}{b}=\overline{hs\theta},\qquad\text{or}\qquad x=a\,\overline{hc\theta},\quad y=b\,\overline{hs\theta},$$

and in ***H'*** $x=a\,\overline{hs\theta}$, $y=b\,\overline{hc\theta}$; the analogy of which to the eccentric Eqs. of the ***E*** is plain. Hyperbolic functions are of some use in higher analysis, and these Eqs. are of interest in Kinematic.

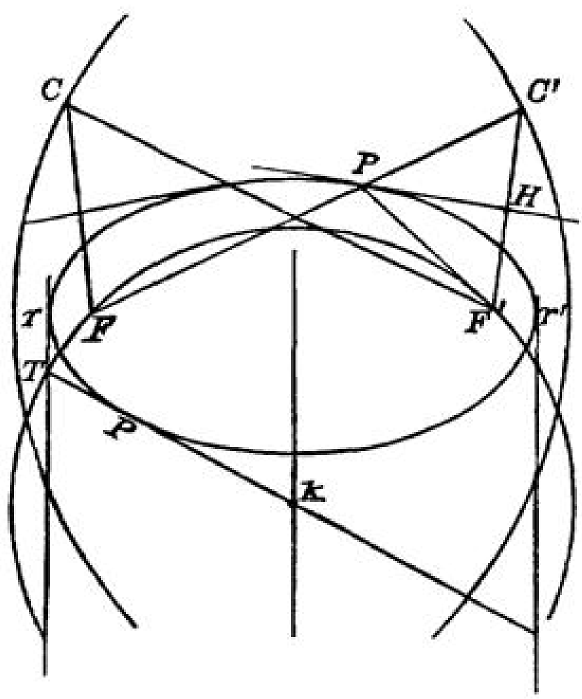
Supplemental Chords.

131. Two chords through a point of an ***E*** or ***H*** and the ends of a diameter are called *supplemental*. They are ∥ *to a pair of conjugate diameters*, for a diameter halving one is clearly ∥ to the other. Hence, if on any diameter of the conic as a chord be described a circle-segment containing a given angle, and a point where this circle-segment cuts the conic be joined to the ends of the diameter, the diameters ∥ to these chords will be conjugate and inclined at the given angle. The problem of drawing conjugate diameters making a given ∡ with each other is thus solved and soluble only when the circle-segment meets the conic in real points. As these points will in general be two, there are in general two pairs of conjugate diameters having a given slope to each other.

Auxiliary Circles.

132. Of these have already been found several, as:

(1) and (2), the *major* and *minor circles* (Art. 107); the major is the locus of the foot of the focal ⊥ on the tangent (Art. 116).



(3) The *director-circle*

$$x^2 + y^2 = a^2 \pm b^2$$

(+ in ***E***, − in ***H***), being the locus of the intersection of pairs of ⊥ tangents (Art. 125).

(4) To these we may now add the two *counter-circles*. If either focal radius of any point of an ***E*** resp. ***H*** be lengthened resp. shortened by the length of the other, the point thus reached will clearly lie on a circle about the first focus, radius $2a$. Since the tangent halves

the angle of the focal radii, the point and the other will be symmetric as to the tangent, or the point will be the *counter-point* of that focus as to the tangent. Hence the locus of the counter-point of either focus is the *counter-circle* about the *other* focus. Also, *the counter-point of either focus* (as to any tangent), *the point of tangence*, and *the other focus* lie on a RL.

(5) The system of *focal circles*. Of these the X-axis is the common power-line, the Y-axis is the centre-line. Any one meets the *tangents through its centre* on the *vertical tangents* (at the ends of the major or real axis). For in the ***E*** the radius of such a circle is $\sqrt{a^2 + b^2 \overline{\cot\epsilon}^2}$, ϵ being the eccentric ∡ of the point of tangence of a tangent through its centre; also the intercept of such a tangent between the vertical tangents is $2\sqrt{a^2 + b^2 \overline{\cot\epsilon}^2}$. Like reasoning holds for the ***H***, $\cot\epsilon$ changing to $\operatorname{cosec}\eta$.

These circles are helpful in problems of construction.

133. In ***P*** the minor circle lies wholly in ∞; the *major* reduces to the *vertical tangent*, the locus of the foot of the focal ⊥ on the tangent (Art. 121); the *director-circle* becomes the *directrix*, the locus of the intersection of pairs of ⊥ tangents (Art. 106); so, too, does the *counter-circle of the focus*, since plainly the *counter-points* of the *focus* lie on the *directrix* (Art. 121). All this the student may also prove analytically by passing to the vertex as origin and reducing the Eqs., remembering that in ***P***, $a = \infty$, $e = 1$, $\frac{2b^2}{a} = 4q$. The focal circles all reduce to the axis of ***P***, since the other focus is at ∞, and so are little useful in construction.

Their place is filled in a measure by the *circles about* △s *circumscribed about* ***P***, all of which *pass through the focus*, as may thus be proved.

Let three tangents touch at P_1, P_2, P_3, and meet by twos at I_3, I_1, I_2; by Art. 114,

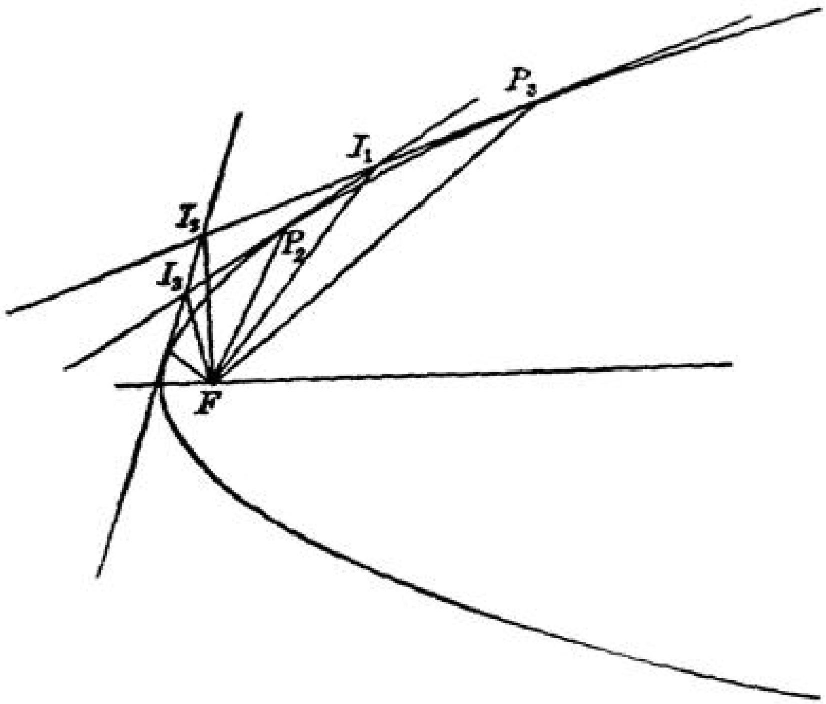
$$∡ P_1FI_3 = ∡ I_3FP_2, \quad \text{and} \quad ∡ P_2FI_1 = ∡ I_1FP_3;$$

whence $\measuredangle I_3FI_1 \equiv \measuredangle I_3FP_2 + \measuredangle P_2FI_1$

$$= \tfrac{1}{2}\measuredangle\{P_1FI_3 + I_3FP_2 + P_2FI_1 + I_1FP_3\}$$

$$= \tfrac{1}{2}\measuredangle P_1FP_3;$$

i.e., the intercept of any tangent between two fixed tangents to P subtends a fixed angle at the focus: *half the* $\measuredangle$ *subtended by the chord through the fixed points of tangence.* Now as the focal $\perp$ on the tangent meets it on the vertical tangent, the slope of the $\perp$ to the axis = the slope of the tangent to the vertical



tangent; but by Art. 121 the former is half the slope of focal radius to the point of tangence; hence *the difference of the slopes of two tangents to the vertical tangent*, i.e., *the* $\measuredangle$ *between two tangents*, is *half the angle between the focal radii to the points of tangence.* On applying this to the case in hand, it appears that the $\measuredangle$s $I_1I_2I_3$ and I_3FI_1 are supplementary; hence the circle about the $\triangle$ $I_1I_2I_3$ goes through F; Q.E.D. Circles circumscribing circumscribed $\triangle$s we may name *focal*.

Vertical Equation of the Conic.

134. If $x - a$ resp. $x + a$ be put for x in the central Eq. of the ***E*** resp. ***H***, the reduced Eq. takes the form

$$y^2 = \frac{2\,b^2}{a}\cdot x - \frac{b^2}{a^2}\cdot x^2 \text{ resp. } y^2 = \frac{2\,b^2}{a}\cdot x + \frac{b^2}{a^2}\cdot x^2,$$

which is therefore the Eq. of the ***E*** resp. ***H*** referred to the axis and the tangent through the left resp. right vertex. If in either we put $\frac{2\,b^2}{a} = 4\,q, \quad a = \infty,$ we get the vertical Eq. of ***P*** $y^2 = 4\,qx$. Now $4\,q$ or $\frac{2\,b^2}{a}$ is the parameter or double ordinate through the focus; hence these Eqs. state the geometric fact that the *square of the ordinate, as compared with the rectangle of parameter and abscissa*, shows: in the Ellipse, *lack;* in the Parabola, *likeness;* in the Hyperbola, *excess.* From this fact the curves seem to have been named. Hence the Eq. of any conic may be brought into the form $y^2 = Rx + Sx^2$, where S is < 0, $= 0$, > 0 resp. for ***E***, ***P***, ***H*** resp.

Lengths of Tangents from a Point to a Conic.

135. These have been found to vary as the $\|$ diameters in the centric conic (Art. 101). In the ***P*** be $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ the points of tangence, t_1, t_2 the tangent-lengths. Then the Cds. of the intersection of the tangents are:

$$x = \frac{y_1 y_2}{4\,q} = \sqrt{x_1 x_2}, \quad y = \frac{y_1 + y_2}{2};$$

i.e., are the *geometric* resp. *arithmetic* means of the Cds. of the points of tangence. Hence, ρ_1 and ρ_2 being the focal radii,

$$t_1^2 = \frac{\overline{y_2 - y_1}^2}{4} + \frac{\overline{y_2 - y_1}^2}{4}\cdot\frac{y_1^2}{4\,q^2} = \frac{\overline{y_2 - y_1}^2}{4}\cdot\left(1 + \frac{x_1}{q}\right)$$

$$= \frac{\overline{y_2 - y_1}^2}{4\,q}(x_1 + q) = \frac{\overline{y_2 - y_1}^2}{4\,q}\cdot\rho_1;$$

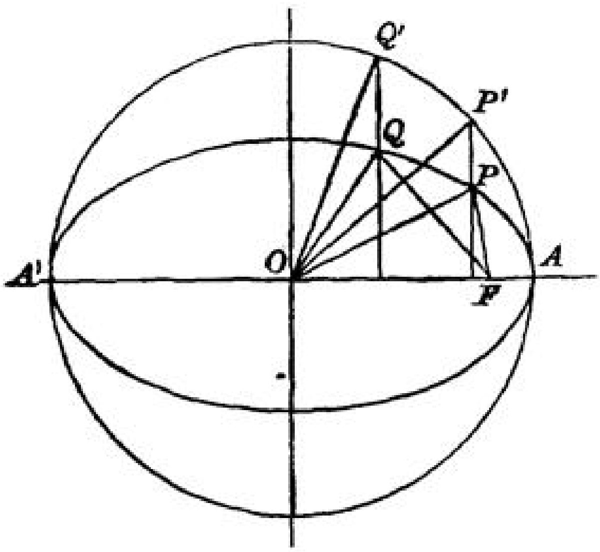
so $$t_2^2 = \frac{\overline{y_2 - y_1}^2}{4\,q}\cdot\rho_2; \quad \therefore\ t_1^2 : t_2^2 = \rho_1 : \rho_2;$$

i.e., *the squared tangent-lengths from a point to a P vary as the focal radii to the points of tangence.*

Areas of Segments and Sectors of a Conic.

136. By Art. 107 the area of an ***E*** is to the area of its major circle as b to a, chords $\perp$ to the axis major being all in this ratio. It is also clear that any segment of the ***E*** is to the corresponding segment of the circle as b to a, and the same holds equally of half-segments reckoned from the axis major.

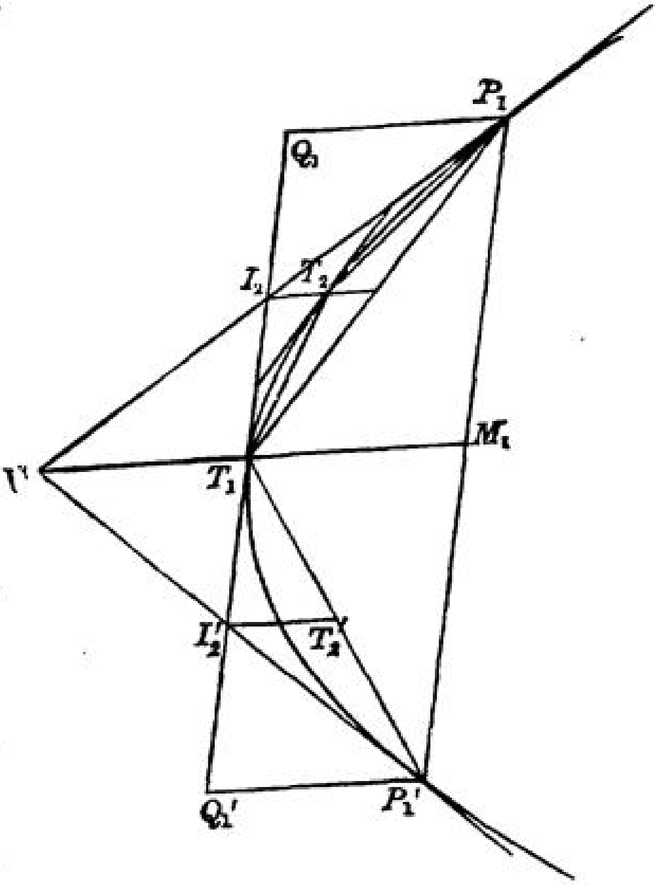
Two corresponding sectors AOP, AOP' are made up of two half-segments in the ratio $b : a$, and two $\triangle$ in the same ratio; hence are themselves in that ratio. So, too, are any other corresponding sectors AOQ, AOQ'; hence so, too, are their differences POQ, $P'OQ'$.



Since the centric $\measuredangle$ AOP or ϕ and the eccentric ϵ or AOP' are connected by the relation $y : x = \tan\phi = \frac{b}{a}\cdot\tan\epsilon$, if the sector be given by its centric $\measuredangle$, we may still use the eccentric.

The area of any focal sector PFQ is the difference of the focal sectors AFQ and AFP, each of which is, again, the difference of a central sector and a $\triangle$.

137. In ***P*** be P_1, P_1' any two points, I_1 the pole of P_1P_1', M_1 the mid-point. Then the RL. I_1M_1 is a diameter, the tract I_1M_1 is halved by P at T_1, the tangent at T_1 is $\parallel$ to P_1P_1', and halves the tangents from I_1 at I_2, I_2'. The $\triangle$s $I_1I_2I_2'$ and $T_1P_1P_1'$ have equal altitudes; the base and therefore the area of the second is twice that of the first. We may proceed with the $\triangle$s $T_1I_2P_1$, $T_1I_2'P_1'$ exactly as with $P_1I_1P_1'$, cutting off by mid-tangents at T_2, T_2' outer areas and by chords to T_2, T_2' inner areas twice as large; and so on without end. The limit of the sum of the outer areas is the outer sector $P_1I_1P_1'$, and the limit of the sum of the inner areas is the inner parabolic segment $P_1T_1P_1'$, cut off by the chord P_1P_1'; hence this latter is twice the former, or $\frac{2}{3}$ of the $\triangle\ P_1I_1P_1'$, or $\frac{2}{3}$ of the parallelogram P_1Q_1' of the chord and the ∥ tangent.



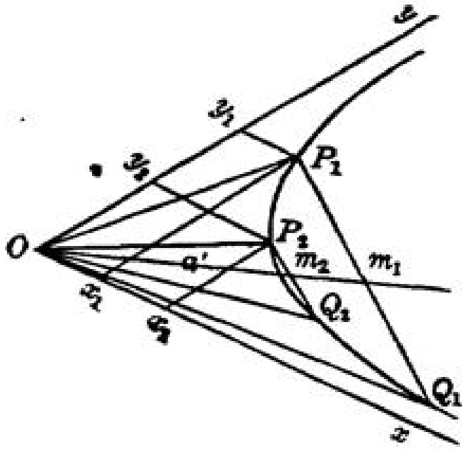
The focal sector from the vertex V to the point $P(x, y)$ is

$$\tfrac{2}{3}xy - \tfrac{1}{2}(x-q)y,$$

or $$\tfrac{1}{6}xy + \tfrac{1}{2}qy.$$

If the axis and the diameter through P cut the directrix at D' and D, the area $VD'DP$ is $\frac{1}{3}xy + qy$, i.e., twice the area of the focal sector. Hence, any *focal sector PFP′ is half the area of the corresponding outer segment between the curve and the directrix and the diameters through P, P′.*

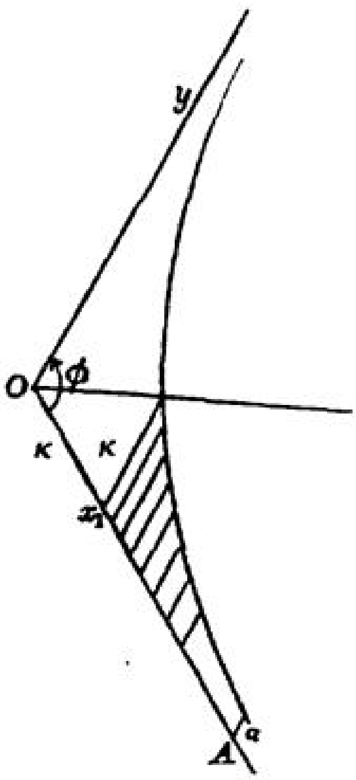
138. If through any two points P_1, P_2 on an ***H*** be drawn ∥s to either asymptote, meeting the other at X_1, X_2, then $P_1X_1X_2P_2$ is called an hyperbolic segment corresponding to the hyperbolic sector P_1OP_2. The $\triangle\ P_1OX_1$, P_2OX_2 are equal, being halves of equal parallelograms (Art. 118); taking each in turn from the figure $P_1OX_2P_2$, we get in turn $P_1X_1X_2P_2$ and P_1OX_1; hence, *corresponding hyperbolic sector and segment are equal.*



If P_1, Q_1 and P_2, Q_2 be ends of two ∥ chords, the sectors P_1OP_2 and Q_1OQ_2 are equal; for the conjugate diameter of the chords halves both the triangular and hyperbolic areas, halving every element of each: taking away equals from equals, we have left the sectors, and therefore their corresponding segments, equal.

If $y = sx + b$ be any chord referred to the asymptotes, then the roots x_1, x_2 of the Eq. $xy \equiv sx^2 + bx = \kappa^2$ are the x's of the ends of the chord; their product, $\frac{-\kappa^2}{s}$, is independent of b, i.e., is the same for all ∥ chords. When the product of the x's is constant, so is the product of the y's by virtue of the relation $xy = \kappa^2$. Hence either set of asymptotic Cds. of the ends of two ∥ chords form a geometric progression. Plainly the converse holds: if any number of like asymptotic Cds., x's or y's, be taken in geometric progression, they will belong to the ends of ∥ chords, and hence the area of sector or segment determined by any two consecutive ones will be constant.

Take now the hyperbolic segment between the ordinates $y = \kappa$ and $y = a$ in the *H* $xy = \kappa^2$, and cut it into n equal sub-segments by ordinates in geometric progression. If r be the ratio, then $a = r^n \kappa$, and $r = \left(\frac{a}{\kappa}\right)^{\frac{1}{n}}$. The end-abscissæ are $x = \kappa$, $x = \frac{\kappa^2}{a}$. The ∥ sides of the first segment are κ and $r\kappa$; its base is $\frac{\kappa}{r} - \kappa$; its area, if ϕ be the asymptotic angle, is $< \kappa\left(\frac{\kappa}{r} - \kappa\right)\sin\phi$. and $> r\kappa\left(\frac{\kappa}{r} - \kappa\right)\sin\phi$. Hence the area S of the whole segment lies between



$$\kappa^2 \sin\phi\, n\left(\frac{1}{r} - 1\right) \text{ and } \kappa^2 \sin\phi\, n(1 - r).$$

The ratio of these extremes is r; if n be taken ever greater and greater, r nears 1, the extremes near each other, keeping S always between them; hence S is the common limit which they both near as n increases without limit. We can readily evaluate $n(1-r)$ or $n\left\{1-\left(\frac{a}{\kappa}\right)^{\frac{1}{n}}\right\}$ for n nearing ∞, by expanding $\left(\frac{a}{\kappa}\right)^{\frac{1}{n}}$. To do this, we write it as a binomial, thus: $\left(1+\overline{\frac{a}{\kappa}-1}\right)^{\frac{1}{n}}$; as a is $<\kappa$, $\frac{a}{\kappa}-1$ is <1; accordingly, the binomial expansion is applicable, being convergent. On expanding, taking from 1, and multiplying by n, there results the series,

$$n\left\{1-\left(\frac{a}{\kappa}\right)^{\frac{1}{n}}\right\}=-\left\{\overline{\frac{a}{\kappa}-1}+\frac{\frac{1}{n}-1}{1\,.\,2}\cdot\overline{\frac{a}{\kappa}-1}^{2}\right.$$

$$\left.+\frac{\overline{\frac{1}{n}-1}\cdot\overline{\frac{1}{n}-2}}{1\cdot 2\cdot 3}\cdot\overline{\frac{a}{\kappa}-1}^{3}+\frac{\overline{\frac{1}{n}-1}\cdot\overline{\frac{1}{n}-2}\cdot\overline{\frac{1}{n}-3}}{1\cdot 2\cdot 3\cdot 4}\cdot\overline{\frac{a}{\kappa}-1}^{4}+\cdots\right\}.$$

As n rises above all limit, $\frac{1}{n}$ sinks below all limit, nears 0, and the numerators become -1, $-1\cdot-2$, $-1\cdot-2\cdot-3$, and the series becomes

$$-\left\{\overline{\frac{a}{\kappa}-1}-\frac{1}{2}\cdot\overline{\frac{a}{\kappa}-1}^{2}+\frac{1}{3}\cdot\overline{\frac{a}{\kappa}-1}^{3}-\frac{1}{4}\cdot\overline{\frac{a}{\kappa}-1}^{4}+\cdots\right\},$$

which, from Algebra, we know to be the negative of the expansion of the natural logarithm of $\frac{a}{\kappa}$ when $0<\frac{a}{\kappa}\leqq 2$, as is the case here. Hence, at last,

$$S=-\kappa^{2}\cdot\sin\phi\cdot\text{nat. log}\frac{a}{\kappa},\quad\text{or}\quad S=\kappa^{2}\sin\phi\cdot\text{nat. log}\frac{\kappa}{a}.$$

The area of a segment whose end-ordinates are a and b, $b<a<\kappa$,

$$\equiv S(a,b)=\kappa^{2}\sin\phi\left\{\text{nat. log}\frac{\kappa}{b}-\text{nat. log}\frac{\kappa}{a}\right\}$$

$$=\kappa^{2}\sin\phi\cdot\text{nat. log}\frac{a}{b}.$$

If κ be taken as linear unit, and if $\phi = 90°$, i.e., if the ***H*** be equilateral, then

$$S = \text{nat. log}\frac{1}{a}, \quad S(a,b) = \text{nat. log}\frac{a}{b};$$

i.e., the area of any segment, reckoned from the vertex, is the natural logarithm of the end-abscissa, and the area of any segment wholly on one side of the vertex is the natural logarithm of the ratio of its end-ordinates. Hence natural logarithms have been called *hyperbolic* logarithms.

Thus far, segments and sectors lie outside of ***H***; but problems about inner ones can now present no difficulty.

Varieties of the Conic.

139. We have found three *species* of conic: ***E***, ***P***, ***H***, according as $-C$ or $h^2 - kj$ is < 0, $= 0$, > 0; but of each there are several varieties, which are now to classify. By Art. 51, on passing to ∥ axes through a new origin (x_1, y_1), the new absolute term becomes

$$c' = (kx_1 + hy_1 + g)x_1 + (hx_1 + jy_1 + f)y_1 + gx_1 + fy_1 + c.$$

If the new origin (x_1, y_1) be the centre of the conic, the coefficients of x_1, y_1 vanish, and the values of x_1, y_1 are $\frac{G}{C}$, $\frac{F}{C}$; hence,

$$c' = gx_1 + fy_1 + c = g\cdot\frac{G}{C} + f\cdot\frac{F}{C} + c\cdot\frac{C}{C} = \frac{\Delta}{C}.$$

Hence the Eq. of the centric conic referred to its centre is

$$kx^2 + 2hxy + jy^2 + \frac{\Delta}{C} = 0.$$

Now, in the ***E***, C or $kj - h^2$ is > 0, hence k and j are like-signed, hence when Δ and they (k and j) are unlike-signed the Eq. is clearly satisfied by real values of x and y, hence the ***E*** is real; but when Δ and they are like-signed the Eq. is satisfied by no real values of x and y, hence the ***E*** is imaginary; also when $\Delta = 0$ no real values satisfy it but the pair (0, 0), hence the Eq. pictures a pair of imaginary RLs. intersecting in the origin (x_1, y_1).

In the ***H***, k and j are unlike-signed and $C < 0$; plainly the Eq. is satisfied by real values of x and y in all cases, but for $\Delta = 0$ it pictures a pair of RLs. through the origin (x_1, y_1).

In case of the non-centric, ***P***, $C = 0$, hence the first three terms of the general Eq. form a perfect square, and we have

$$(\sqrt{k}\,x \pm \sqrt{j}\,y)^2 + 2\,gx + 2\,fy + c = 0.$$

Solved as to the parenthesis, this Eq. takes the form

$$\sqrt{k} \cdot x \pm \sqrt{j} \cdot y + \frac{f}{\sqrt{j}} = \pm \frac{1}{\sqrt{j}} \sqrt{2\,Gx - K}.$$

Accordingly, the Eq. pictures a real ***P*** save when $G = 0$. Then it pictures two RLs., which are *always* ∥, and are real and separate, coincident, or imaginary, according as $K < 0$, $= 0$, or > 0. If $j = 0$, like conclusions hold on changing G to F and K to J.

Hence the following table:

$C > 0$	$k\Delta < 0$	real ellipse.
	$\Delta = 0$	two imaginary RLs.
	$k\Delta > 0$	imaginary ellipse.
$C = 0$	$\Delta < 0$	real parabola.
	$\Delta = 0$	two parallel RLs.
	$\Delta > 0$	real parabola.
$C < 0$	$k\Delta < 0$	real hyperbola.
	$\Delta = 0$	two real RLs.
	$k\Delta > 0$	real hyperbola.

CHAPTER VII.

SPECIAL METHODS AND PROBLEMS (Continued).

Determination and Construction of the Conic.

140. The Eq. of the conic has six constants, but division by any one reduces the number to five, which are therefore the independent arbitraries of the Eq. Hence five independent simple conditions are needed and enough to determine a conic, since they determine the five arbitraries. Conditions are *independent* when no one can be drawn out from the others, *simple* when each fixes but *one* relation among the arbitraries. Such are that the conic shall go through certain points or touch certain RLs. *A multiple* condition fixes more than one relation among the arbitraries. Thus, that the conic *touch a certain RL. at a certain point* is a *double* condition, namely, that the conic pass through two given consecutive points; that the *centre* be (x_1, y_1) is a *double* condition, for it makes

$$kx_1 + hy_1 + g = 0 \quad \text{and} \quad hx_1 + jy_1 + f = 0;$$

that a *given direction* be *asymptotic* is *simple*, since it fixes one point at ∞, but that a *given RL.* be an *asymptote* is *double*, since it fixes two points at ∞; that a *given point* be *focus* is *double*, since three tangents besides would determine the conic by determining three points of the major circle; to give the direction of an axis is to give a point (at ∞) in $\boldsymbol{P}$, or to give the relation $\dfrac{2h}{k-j}$ = a constant in $\boldsymbol{E}$ and $\boldsymbol{H}$, a *simple* condition; hence, to give both axes in position, since this fixes the centre besides, is to impose a *triple* condition. A given eccentricity is in general one simple condition, but if $e = 0$, the conic is a circle, which implies the two conditions of Art. 58. For exercises, see pp. 210–213.

141. The case of a conic fixed by five points merits special attention. If the five lie on a RL., that RL. and any other are the conic; there is a *two*-fold ∞ of solutions in pairs of RLs. If only four lie on a RL., that RL. and any other through the fifth are the conic; there is a *one*-fold ∞ of solutions in pairs of RLs. If only three lie on a RL., that RL. and the RL. through the other two are the conic; there is but *one* solution. Thus far the solutions have been pairs of RLs., since no curve-conic has three points on a RL. If only two points lie on a RL., the Eq. of the conic is got by assuming

$$kx^2 + 2\,hxy + jy^2 + 2\,gx + 2\,fy + c = 0,$$

which with five like Eqs. got by replacing the current Cds. by the Cds. of the five points will form a system of six Eqs. homogeneous of first degree in the six unknowns, k, h, j, g, f, c; since the absolutes are all 0, the condition of consistence is that the determinant of the coefficients of the unknowns vanish; hence that determinant equated to 0 is the Eq. sought. But to avoid the tedium of reducing this determinant, we may proceed better thus:

Be $L_{12}=0$, $L_{23}=0$, $L_{34}=0$, $L_{41}=0$, $L_{13}=0$, $L_{24}=0$ the six RLs. fixed by four of the points; to denote that in any L the current Cds. are replaced by the Cds. of the fifth point, prefix the superscript 5. Then

$$L_{12}\cdot L_{34}=0 \quad \text{and} \quad L_{23}\cdot L_{41}=0$$

are a pair of pairs of RLs., i.e., a pair of conics through the four points; hence $L_{12}\cdot L_{34}-\lambda L_{23}\cdot L_{41}=0$ is a conic, being of second degree, through the four points. If this conic goes through the fifth point, then ${}^5L_{12}\cdot {}^5L_{34}-\lambda\, {}^5L_{23}\cdot {}^5L_{41}=0$; whence finding the value of λ and putting it for λ in the other Eq. we get the Eq. of the conic sought.

Since we can find a fit value of λ for any fifth point, it follows that $L_{12}\cdot L_{34}-\lambda\, L_{23}\cdot L_{41}=0$ represents a conic passing through the four points and any other point. Also a conic through four points of which no three are on a RL. can sustain but one more condition, and λ may be taken to satisfy any one condition; hence the above Eq. represents the whole system of conics through four points; and since for any fifth point there is found but one value of λ, it is seen that *through five points, no three of which lie on a RL., one, and only one, conic passes.*

142. Two special forms of this Eq. of a system of conics are specially useful.

(1) Take two of the L's for axes; then their Eqs. are $x = 0$, $y = 0$; the Eqs. of the other two are

$$lx + my + 1 = 0 \quad \text{and} \quad l'x + m'y + 1 = 0,$$

and the Eq. of the system is

$$(lx + my + 1)(l'x + m'y + 1) - \lambda xy = 0.$$

The terms of second degree are

$$ll'x^2 + (lm' + l'm - \lambda)xy + mm'y^2.$$

They form a square, i.e., the conic is a ***P***, when, and only when,

$$(lm' + l'm - \lambda)^2 = 4\, ll'mm'.$$

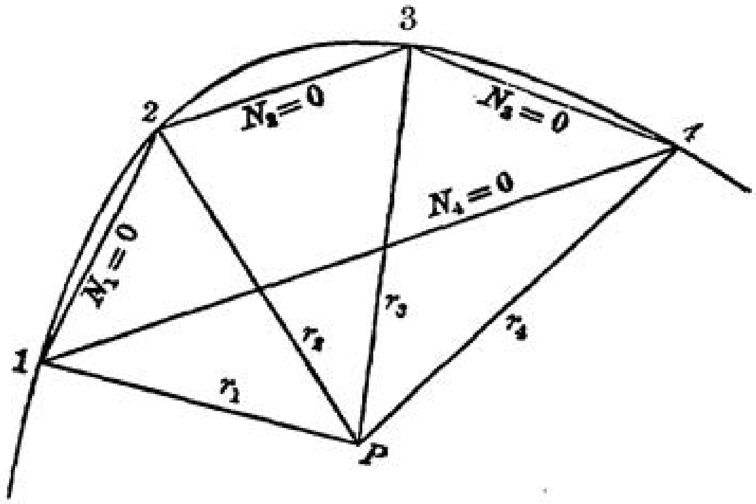
This Eq. is of second degree in λ; hence *two, and only two, of a system of conics through four points are **P**'s.* The student may easily investigate the conditions under which the ***P***'s are real or imaginary, separate or coincident.

(2) Take the Eqs. of the RLs. in the N. F., and drop the second subscript; then $N_1 \cdot N_3 - \lambda N_2 \cdot N_4 = 0$ is a conic about the 4-side whose counter-sides are $N_1 = 0$, $N_3 = 0$ and $N_2 = 0$, $N_4 = 0$. Call the lengths of the sides, i.e., the chords of the conic, c_1, c_2, c_3, c_4. From any point P draw rays r_1, r_2, r_3, r_4, to the vertices of the 4-side. Then the double areas of the Δs are

$$N_1 \cdot c_1 = r_1 \cdot r_2 \cdot \sin \widehat{r_1 r_2}, \quad N_2 \cdot c_2 = r_2 \cdot r_3 \cdot \sin \widehat{r_2 r_3},$$

$$N_3 \cdot c_3 = r_3 \cdot r_4 \cdot \sin \widehat{r_3 r_4}, \quad N_4 \cdot c_4 = r_4 \cdot r_1 \cdot \sin \widehat{r_4 r_1};$$

whence
$$\frac{N_1 \cdot N_3}{N_2 \cdot N_4} \cdot \frac{c_1 \cdot c_3}{c_2 \cdot c_4} = \frac{\sin \widehat{r_1 r_2} \cdot \sin \widehat{r_3 r_4}}{\sin \widehat{r_2 r_3} \cdot \sin \widehat{r_4 r_1}}.$$



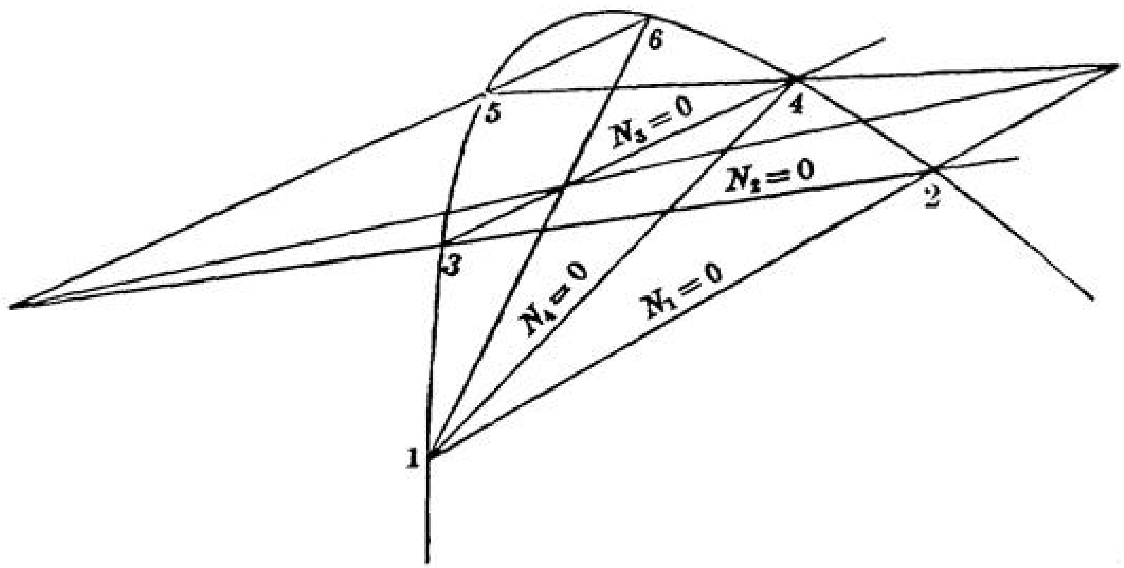
The right member of this Eq. is the cross-ratio of the four rays, and the left is constant so long as $N_1 \cdot N_3 : N_2 \cdot N_4$ is; i.e., so long as the centre of the pencil, P, is on a conic through the four points. Hence the *cross-ratio of a pencil from any point of a conic through four fixed points of the conic is constant.*

143. If now we take any sixth point on the conic, we shall have an inscribed 6-side. The sides and vertices numbered 1 and 4, 2 and 5, 3 and 6, i.e., whose numbers differ by 3, are called *counter*. Let the Eqs. of the sides of the primary 4-side stand as in Art. 142; then the Eq. of the fourth side of the 6-side, i.e., the side from 4 to 5, will be $N_3 - \kappa N_4 = 0$, since it is a ray of the pencil (N_3, N_4); so, too, the sixth side, from 6 to 1, will have for its Eq. $N_1 - \kappa' N_4 = 0$. Since 5 is on the conic $N_1 N_3 - \lambda N_2 N_4 = 0$, we have

$$\kappa = \frac{N_3}{N_4} = \lambda \frac{N_2}{N_1}, \quad \text{or} \quad \frac{\kappa}{\lambda} = \frac{N_2}{N_1};$$

hence the ray from 2 to 5 is $\kappa N_1 - \lambda N_2 = 0$. By like reasoning, the ray from 3 to 6 is $\kappa' N_3 - \lambda N_2 = 0$. Hence the fifth side is the common ray of the two pencils

$$N_3 - \kappa N_4 + \mu(\kappa N_1 - \lambda N_2) = 0, \quad N_1 - \kappa' N_4 + \mu'(\kappa' N_3 - \lambda N_2) = 0.$$



These Eqs. are the same when $\mu = 1 : \kappa'$, $\mu' = 1 : \kappa$; hence the fifth side is $\kappa N_1 - \lambda N_2 + \kappa' N_3 - \kappa\kappa' N_4 = 0$. The RL. through the intersections of *counter* sides $N_1 = 0$ and $N_3 - \kappa N_4 = 0$, $N_3 = 0$ and $N_1 - \kappa' N_4 = 0$ is the common ray of the pencils

$$N_1 - r(N_3 - \kappa N_4) = 0, \quad N_1 - \kappa' N_4 - r' N_3 = 0,$$

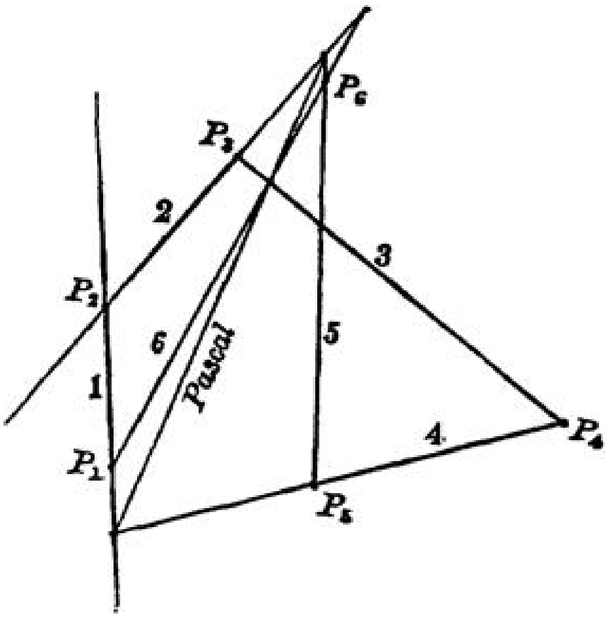
i.e., it is the RL. $\kappa N_1 - \kappa\kappa' N_4 + \kappa' N_3 = 0$. But this RL. goes through the intersection of the third pair of counter sides $N_2 = 0$ and $\kappa N_1 - \kappa\kappa' N_4 + \kappa N_3 - \lambda N_2 = 0$. Hence we have

Pascal's Theorem. *The three pairs of counter sides of a 6-side inscribed in a conic intersect on a RL.* called a **Pascal** (RL.). Since six points may be thought arranged circularly in 5! or 120 ways, and since a ray joining two points may be counted in two ways, it follows that there are sixty really different orders, and so sixty really different Pascal RLs.

Pascal discovered this beautiful relation, and built upon it a theory of the conic.

144. We have seen (Art. 141) how to form the Eq. of a conic through five points; Pascal's Theorem enables us to construct it without knowing its Eq., and more rapidly than by Art. 71, thus:

Draw through 5 any RL.; it will cut the conic in 6; find the intersections of sides 1 and 4, 2 and 5 (the ray just drawn); through them draw the Pascal RL.; it will pass through the intersection of sides 3 and 6; from 1 draw side 6 through the intersection of the Pascal and side 3; it will meet side 5 in point 6. So we may find any number of points of the conic.



If side 5 be drawn ∥ to side 1, the RL. halving the tracts $\overline{1\,2}$ and $\overline{5\,6}$ will be a diameter of the conic; a second diameter can be got by drawing side 5 ∥ to side 3; these two diameters fix the conic's centre.

If side 5 is to touch the conic at 5, point 6 must fall on 5; hence, draw side 6 from 1 through 5; through the intersections of the counter-sides 1 and 4, 3 and 6, draw the Pascal; it will cut side 2 in a point of side 5; the RL. through this point and 5 will touch the conic at 5. So we may draw any number of tangents to the conic.

145. We have learned to construct the conic by points and form its Eq. from given conditions; it remains to determine its elements, *centre*, *axes*, *foci*, *directrices*, *asymptotes*, from its Eq.

By Art. 93 the centre is the point $(G:C, F:C)$. If C be >0 resp. <0, the Cds. are finite, the conic is an ***E*** resp. ***H***.

Transformed to ∥ axes through the centre, the Eq. becomes

$$kx^2 + 2hxy + jy^2 + \frac{\Delta}{C} = 0.$$

The asymptotes are now

$$kx^2 + 2hxy + jy^2 = 0;$$

hence, if we pass back to the old origin and axes, the Eq. of the asymptotes becomes

$$kx^2 + 2hxy + jy^2 + 2gx + 2fy + c - \frac{\Delta}{C} = 0;$$

i.e., *the Eqs. of the conic and its asymptotes differ only by the constant* $\frac{\Delta}{C}$

The axes halve the ∡s between the asymptotes; hence they are the RLs. $x^2 - y^2 = \frac{k-j}{h}xy$ by Art. 53. As $\tan 2\theta = \frac{2h}{k-j}$, θ being the slope of an axis to the X-axis, we may now find the directions of the axes by drawing the RL. $y = \frac{2h}{k-j}x$ and halving its slope to the X-axis; but to determine the lengths of the axes is still tedious. A guide to a simpler solution of both problems in one is the reflection that the diameters of a conic through its intersections with a concentric circle are equal, being diameters of the circle, hence are like-sloped to either axis of the conic; accordingly, when these diameters fall together, it is on an axis of the conic. Now by Arts. 30, 50

$$kx^2 + 2hxy + jy^2 + \frac{\Delta}{C} \cdot \frac{x^2 + y^2}{r^2} = 0$$

is a pair of RLs. through the intersections of the conic with the concentric circle $x^2 + y^2 = r^2$, and the origin (centre). They fall together when the Eq. is a square, i.e., when

$$\left(k + \frac{\Delta}{Cr^2}\right)\left(j + \frac{\Delta}{Cr^2}\right) = h^2.$$

The roots r_1^2, r_2^2 of this Eq. are therefore the *squared half-axes* of the conic; putting them in turn for r^2 in the Eq. of the pair and taking the second root, we get the *Eqs. of the axes.* The axes thus found in size and position, the foci are found by laying off on the proper axis the proper focal distance, $ae = \sqrt{a^2 \mp b^2}$. The directrices are easily constructed as polars of the foci.

To find the eccentricity, suppose the Eq. of the curve referred to its axes to be $Ax^2 + By^2 = 1$; then, as A and B are the reciprocals of the squared half-axes, we have $A = B(1 - e^2)$. Now pass back to the original central Eq.

$$kx^2 + 2\,hxy + jy^2 + \frac{\Delta}{C} = 0.$$

By Art. 102, since $\omega = 90°$,

$$A + B = k + j \quad \text{and} \quad AB = kj - h^2.$$

On elimination of A and B from the three Eqs., there results

$$e^4 + \frac{(k-j)^2 + 4\,h^2}{kj - h^2}\,(e^2 - 1) = 0.$$

This Eq. teaches that there are *two pairs* of counter-eccentricities of a centric conic, corresponding to the two pairs of foci, one real, one imaginary; if e_1^2, e_2^2 be the squares,

$$e_1^2 + e_2^2 = e_1^2 \cdot e_2^2 = -\frac{(k-j)^2 + 4\,h^2}{kj - h^2}.$$

Hence $\dfrac{1}{e_1^2} + \dfrac{1}{e_2^2} = 1$, the *sum of the reciprocals of the squared eccentricities* is 1. Also, if $kj - h^2 > 0$, the product of e_1^2 and e_2^2 is $-$; i.e., the squared eccentricities are one $+$, one $-$; i.e.. in ***E*** *one* pair of eccentricities are real, *one* imaginary. If $kj - h^2 < 0$, both the sum and the product of the squared eccentricities are $+$; hence both squares are $+$, hence in ***H*** *both* pairs of eccentricities are *real.* The *real* foci lie on the *real* axis, which latter falls on the X- resp. Y-axis when A resp. B is $+$, and the corresponding squared eccentricity is

$$\frac{B-A}{B} \quad \text{resp.} \quad \frac{A-B}{A}.$$

146. The Eq. of ***P***, as the highest terms form a square, may be written

$$(\kappa x + \iota y)^2 + 2\,gx + 2\,fy + C = 0.$$

The RL. $\kappa x + \iota y = 0$ is a diameter (Art. 97), and the RL. $2\,gx + 2\,fy + C = 0$ is a tangent at its end, since on combining the last Eq. with the Eq. of the ***P*** there result two equal pairs of values of x and y, which also satisfy $\kappa x + \iota y = 0$. This diameter and tangent are not in general ⊥, since not in general is $\frac{g\kappa}{f\iota} = -1$. But by adding and subtracting $2\,\kappa\lambda + 2\,\iota\lambda + \lambda^2$ we may write the Eq. of the ***P*** thus:

$$(\kappa x + \iota y + \lambda)^2 + 2(g - \kappa\lambda)x + 2(f - \iota\lambda)y + c - \lambda^2 = 0.$$

The RL. $\kappa x + \iota y + \lambda = 0$ is still a diameter, being ∥ to $\kappa x + \iota y = 0$, and $2(g - \kappa\lambda)x + 2(f - \iota\lambda)y + c - \lambda^2 = 0$ is the tangent at its end for the same reason as before; they are ⊥ if

$$-\frac{\kappa}{\iota} = \frac{f - \iota\lambda}{g - \kappa\lambda}, \text{ i.e., when, and only when, } \lambda = \frac{\kappa f + \iota g}{\kappa^2 + \iota^2}.$$

Hence when λ has this value, the two RLs. are resp. *axis* and *vertical tangent* of the ***P***, and their intersection is the *vertex*.

The parameter $4\,q$ is the ratio of the squared distance of a point of ***P*** from the axis to its distance from the vertical tangent; i.e.,

$$4\,q = \frac{(\kappa x + \iota y + \lambda)^2}{\kappa^2 + \iota^2} : \frac{2(g - \kappa\lambda)x + 2(f - \iota\lambda)y + c - \lambda^2}{2\sqrt{(g - \kappa\lambda)^2 + (f - \iota\lambda)^2}}$$

$$= \frac{2\sqrt{(g - \kappa\lambda)^2 + (f - \iota\lambda)^2}}{\kappa^2 + \iota^2},$$

whence on substitution for λ,

$$4\,q = \frac{2(\kappa g - \iota f)}{(\kappa^2 + \iota^2)^{\frac{3}{2}}} = \frac{2(\sqrt{k}g - \sqrt{j}f)}{(k + j)^{\frac{3}{2}}}.$$

The exponent $\frac{3}{2}$ leaves the sign of $4q$ at will, as it should be, since the + direction of the X-axis is yet at will. The ***P*** clearly lies on the − side of

$$2(g-\kappa\lambda)x+2(f-\iota\lambda)y+c-\lambda^2=0.$$

The focus is on the axis q distant from the vertex; or we may determine its Cds. by finding the intersection of the tangent $2gx+2fy+c=0$ and the vertical tangent, and through it drawing a ⊥ to the former, which will cut the axis at the focus.

147. When the elements are found, the curve may be actually traced by the methods of *points*, of *tangents*, or of *continuous motion*.

(1) Cut the major axis of an ***E*** anywhere within the foci; with the two parts as radii draw circles about the foci; each pair of circles, the sum of whose radii is the axis major, intersect in two points of the ***E*** (Art. 101). Like holds for the ***H*** on changing *within* to *without* and *sum* to *difference*.

(2) From the end of any radius of the major circle of an ***E***, drop a ⊥ on the axis major; from the intersection of the radius with the minor circle draw a ∥ to the axis major; it will cut the ⊥ in a point of the ***E***; for it cuts it in the ratio $b:a$.

From any point on the real axis of an ***H*** draw a tangent-tract to the major circle; also draw a ∥ tangent-tract to the minor circle; at the point lay off the equal of the second tangent-tract ⊥ to the axis; its end is a point of the ***H***; for the first tangent-tract is clearly $a\tan\eta$, and the second is $b\tan\eta$ or y.

(3) Through a focus draw chords of the major circle of the ***E*** resp. ***H***; through their ends draw ⊥s to them; the ⊥s touch ***E*** resp. ***H***; for the feet of focal ⊥s on the tangents lie on the major circle.

(4) Join cross-wise the intersections of a focal circle with the vertical tangents to an ***E*** resp. ***H***; the junction-lines will touch the ***E*** resp. ***H***, by Art. 132.

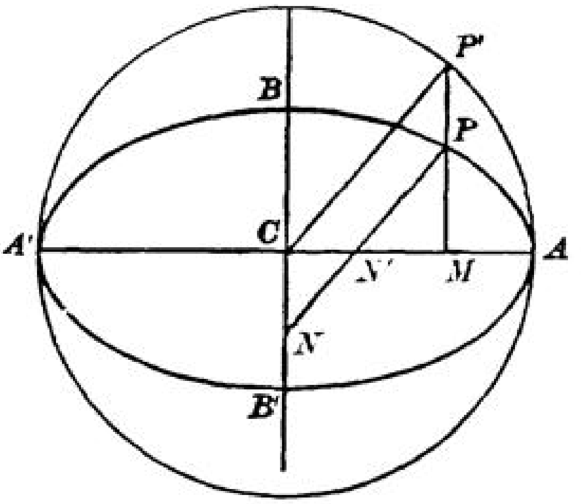
By either (3) or (4), more easily by (4), enough tangents may soon be drawn to shadow forth the curve quite clearly.

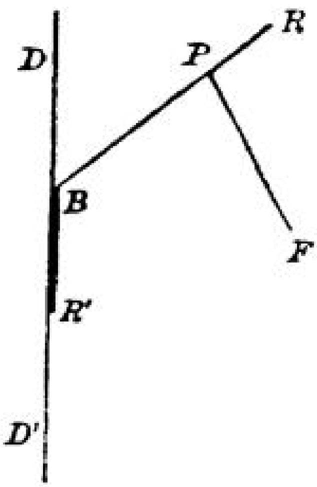
(5) If the ends of a string $2a$ long be fastened less than $2a$ apart, and it be stretched by a sliding pencil, this will trace an ***E*** whose *axis major* is the *length*, whose *foci* are the *ends*, of the string.

(5′) If the ends of a string be fastened, one at a point, the other at the end of a ruler $2a$ longer than the string, and the string be kept stretched against the ruler by a pencil while the ruler turns about its other end fastened at a second point more than $2a$ apart from the first, the pencil-point will move on an ***H*** whose *foci* are the *fixed points* and whose *real axis* is $2a$.

These constructions rest on the same properties as those of (1).

(6) If one end N of a ruler a long slide on a fixed bar BB' while a fixed point N' of the ruler slides along a $\perp$ bar AA', the other end P of the ruler will trace an ***E*** whose axis major is $2\,NP = 2a$, whose axis minor is $2\,N'P = 2b$. For draw a circle with radius a about the cross-point C, and a radius CP' $\parallel$ to NP. Then $MP : MP' = b : a$. The three bars form a pair of *elliptic compasses*.





(6′) If a ruler RBR' bent at B slide along a straight-edge DD' while a pencil-point P keeps a string RB long stretched against the ruler, one end of the string being fastened at R, the other at a fixed point F, then P will trace an ***H***, of which F is a focus, DD' a directrix, BR an asymptotic direction. For the distances of P from F and DD' are clearly in a fixed ratio, namely, $\sec\theta$, where

$\theta = \measuredangle RBR' - 90°$; hence $e = \sec\theta$, $\frac{1}{e} = \cos\theta$, and $\cos^{-1}\frac{1}{e}$ is the slope of an asymptote to the X-axis.

Among constructions of ***P*** by points this seems simplest:

(7) About the focus with any radius $> q$ draw a circle; from where it cuts the axis lay off $2q$ toward the focus; through the point thus reached draw a ⊥ to the axis; it will cut the circle in points of P; also the junction-lines of the points of P and the other end of the diameter will touch the ***P*** (Art. 121).

(8) From the focus draw any ray to the vertical tangent; through their intersection draw a ⊥ to the ray; it will touch ***P***; also a second focal ray having twice the slope of the first to the axis will meet the tangent at the point of tangence (Art. 121).

Or, the point of touch may be found by remembering that the focal ⊥ on the tangent halves the tangent-tract from the axis. Constructions of ***P*** by points and by tangents involve each other.

(9) Construction (6′) of ***H*** will yield ***P*** when the ruler is bent at right angles; for then $\theta = 0$, $e = 1$.

Confocal Conics.

148. *Confocal* conics are clearly also *co-axial*, for the foci fix the axes in position; if $2a$, $2b$ and $2a_1$, $2b_1$ be their axes, then $a^2 - b^2 = a_1^2 - b_1^2 =$ *squared central distance of focus.*

Conversely, *if two conics be co-axial and* $a^2 - b^2 = a_1^2 - b_1^2$, they are *confocal*, the foci clearly falling together.

On transposing, we get $a^2 - a_1^2 = b^2 - b_1^2$; accordingly we may put $a^2 + \lambda$ for a_1^2, at the same time putting $b^2 + \lambda$ for b_1^2.

Hence $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ is a family of confocal conics, and all conics confocal with the base conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are got by letting λ range from $-\infty$ to $+\infty$.

For $-\infty < \lambda < -a^2$, the conics are imaginary; for $\lambda = -a^2$, the conic is a pair of RLs. fallen together on the Y-axis; for

$-a^2 < \lambda < -b^2$, the conics are ***H***'s sinking down to the X-axis; for $\lambda = -b^2 - 0$, the conic is the doubly-laid X-axis *without* the foci; for $\lambda = -b^2 + 0$, it is the doubly-laid X-axis *within* the foci; for $-b^2 < \lambda < \infty$, the conics are ***E***'s swelling out from the X-axis toward the concentric circle with ∞ radius.

The Eq. $\dfrac{x^2}{a^2+\lambda} + \dfrac{y^2}{b^2+\lambda} - 1 = 0$ is of second degree in λ; hence to any pair (x, y) correspond two values of λ; i.e., *through any point in the plane pass two, and only two, confocals.* For λ very great and $+$, the quadratic is $-$; for λ nearing $-b^2$, it is $+$; for $\lambda = -b^2$, it springs from $+\infty$ to $-\infty$; for λ nearing $-a^2$, it is again $+$; for $\lambda = -a^2$, it again springs from $+\infty$ to $-\infty$, and thence stays $-$. Accordingly, the quadratic changes sign by passing through 0 only for λ between $+\infty$ and $-b^2$, and for λ between $-b^2$ and $-a^2$; i.e., the two confocals which pass through any point are the one an ***E***, the other an ***H***.

The tangent and normal of the ***E*** halve the outer resp. inner ∡s of the focal radii to the point, the normal and tangent of the ***H*** halve the same ∡s; hence they are the same pair of RLs., the normal of one curve is the tangent of the other; hence *confocals cut each other, wherever they cut, orthogonally.*

149. *Confocal curves* (and especially *confocal surfaces*) form a system of *rectang. Cd. lines* (and *surfaces*) of great import to Higher Geometry and Mechanics. It is in place here only to show how *ordinary Cds.* may be expressed through *confocal Cds.: pairs of values* of λ yielding *confocals* through the point.

Be $\dfrac{x^2}{a^2+\lambda_1} + \dfrac{y^2}{b^2+\lambda_1} = 1$ and $\dfrac{x^2}{a^2+\lambda_2} + \dfrac{y^2}{b^2+\lambda_2} = 1$ the confocals.

Multiply by the divisors of y^2 and subtract; there results

$$x^2 = (\lambda_1 - \lambda_2) : \left\{ \frac{b^2+\lambda_1}{a^2+\lambda_1} - \frac{b^2+\lambda_2}{a^2+\lambda_2} \right\}$$

$$= (\lambda_1 - \lambda_2)(a^2+\lambda_1)(a^2+\lambda_2) : (\lambda_1 - \lambda_2)(a^2 - b^2)$$

$$= \frac{(a^2+\lambda_1)(a^2+\lambda_2)}{a^2 - b^2},$$

and on exchanging a^2 and b^2,

$$y^2 = -\frac{(b^2+\lambda_1)(b^2+\lambda_2)}{a^2-b^2}.$$

Hence $\quad x^2+y^2 = a^2+b^2+\lambda_1+\lambda_2 = \overline{a^2+\lambda_1}+\overline{b^2+\lambda_2}$

$$= \overline{a^2+\lambda_2}+\overline{b^2+\lambda_1} = r^2.$$

This r being regarded as a half-diameter of the confocal λ_1, its conjugate r_1' is given by the Eq.

$$r_1'^2 = \overline{a^2+\lambda_1+b^2+\lambda_1} - \overline{a^2+\lambda_1+b^2+\lambda_2}$$

$$= \lambda_1 - \lambda_2;$$

and so $\quad r_2'^2 = \lambda_2 - \lambda_1;$

whence $\quad r_1'^2 + r_2'^2 = 0;$

i.e., the conjugates to a common diameter of two confocals are alike in size, one real, one imaginary. Also, by Art. 111, if p_1 resp. p_2 be the central $\perp$ on the tangent to λ_1 resp. λ_2 $\parallel$ to r_1' resp. r_2', or through the end of r, then

$$p_1^2 = \frac{\overline{a^2+\lambda_1}\cdot\overline{b^2+\lambda_1}}{\lambda_1-\lambda_2}, \text{ and } p_2^2 = \frac{\overline{a^2+\lambda_2}\cdot\overline{b^2+\lambda_2}}{\lambda_2-\lambda_1}.$$

Thus may all geometric elements of the system of confocals be expressed through the parameters λ_1, λ_2, and their relations studied.

Similar Conics.

150. From Art. 82 it is clear that any conic K_1 similar to the central conic K must be congruent with some conic K_2 concentric with K and similar to it; K_1 and K_2 differ only in that K_1 has been *pushed* and *turned* as to K. Since the centres of K and K_2 fall together in the centre of similitude, the centres of K and K_1 are corresponding points or centres of similarity.

As K_1 and K_2 are congruent, in dealing with their metric relations we may put the one for the other. Now the eccentricity of K is $\{\overline{a^2-b^2} : a^2\}^{\frac{1}{2}}$, that of K_2 or K_1 is $\{\overline{a_1^2-b_1^2} : a_1^2\}^{\frac{1}{2}}$;

in K and K_2, a and a_2 (or a_1) correspond, so do b and b_1; hence,

$$\frac{a}{a_1}=\frac{b}{b_1}, \quad \text{or} \quad \frac{a}{b}=\frac{a_1}{b_1};$$

hence *the eccentricities of similar conics are equal.* Conversely, if $e=e_2=e_1$, or if

$$\frac{a^2-b^2}{a^2}=\frac{a_1^2-b_1^2}{a_1^2},$$

by decomposing, we get

$$\frac{a}{a_2}=\frac{b}{b_2},$$

or the conics are similar.

Hence *eccentricities equal* is the *necessary* and *sufficient* condition of *similarity* in central conics.

As $\boldsymbol{P}$ is simply a central conic whose centre has retired to ∞ while the parameter has stayed finite, we may at once infer that all $\boldsymbol{P}$'s are similar, having the same eccentricity $e=1$, and all conics similar to a $\boldsymbol{P}$ are $\boldsymbol{P}$'s. Or we may place the $\boldsymbol{P}$'s vertex on vertex, axis on axis; then their Eqs. will be $y^2=4\,qx$, $y^2=4\,q_1x$; or $\overline{\rho\sin\theta}^2=4\,q\rho\cos\theta$, $\overline{\rho_1\sin\theta_1}^2=4\,q_1\rho_1\cos\theta_1$; hence for $\theta=\theta_1$, $\rho:\rho_1=q:q_1$; or the $\boldsymbol{P}$'s are similar, the *ratio of similitude* being the *ratio of their parameters.*

Since in similar conics the eccentricities are equal, the expression $(2-e^2):\overline{1-e^2}$ must be the same in them; hence, by Art. 145, $(k+j)^2:\overline{kj-h^2}$ must be constant and equal

$$\overline{k_1+j_1}^2:\overline{k_1j_1-h_1^2}.$$

This, then, is the *condition that K and K_1, given by their Eqs., be similar.* For oblique axes we must write

$$k+j-2\,h\cos\omega$$

for $k+j$, as is plain from Art. 102.

If now K and K_1 be *like-placed*, i.e., have corresponding tracts ||, the Eq. of K_2 becomes that of K_1 on change of origin only; by such change k_1, h_1, j_1 are not changed, hence they are the same in the Eqs. of K_1 and K_2; but in K and K_2 the intercepts on the axes, origin being at centre, vary inversely as $\sqrt{k}$, $\sqrt{k_1}$ (or $\sqrt{k_2}$), $\sqrt{j}$, $\sqrt{j_1}$; and since these tracts correspond, $k:k_1=j:j_1=r^2$; and since $\frac{(k+j)^2}{kj-h^2}$ is constant, we have also $\frac{h}{h_1}=r^2$.

In two ***P***'s like-placed, or with ‖ axes, it is plain that $\frac{k}{j}=\frac{k_1}{j_1}$, since each is the squared direction-coefficient of the axis; hence in all cases the triple equality $\frac{k}{k_1}=\frac{h}{h_1}=\frac{j}{j}$ shows that the *two conics are similar and similarly placed.*

The Conic as the Projection of a Circle.

151. By Art. 107 the ***E*** is a ‖ projection of a circle, the only kind of projection yet spoken of. The notion may be widened thus:

The point P' where any plane Π' cuts any ray from S is called the **central** *projection of any point P of the ray, on the plane.* S is called the *centre* of projection, Π' the *plane* of projection. Only when S is at ∞ does *central* pass over into ‖ projection.

Rays from S through points on a RL. lie in a plane; hence their intersections with Π' lie on a RL.; i.e., *the projection of a* RL. *is a* RL.

If this RL. cut a plane curve in n points, its projection will cut the projection of the curve in n points, the projections of the original n points; since n fixes the degree of the curve, that *degree is unchanged by projection;* e.g., *the projection of a conic is a conic.*

If the points in which the RL. meets the curve be consecutive, so will be their projections; hence, *the projection of a tangent to a curve is tangent to the projection of the curve.*

Hence, too, the *projections* of *pole* and *polar* as to a conic are *pole* and *polar* as to the *projection* of the conic; also, *projections* of *conjugates*, points or RLs., are *conjugate*.

All points of a plane through the centre ‖ to Π', and no others, are projected into ∞; hence, to project a RL. into ∞, *project it on a plane* ‖ *to the plane through it and the centre.*

A *pencil of rays* is clearly projected into a *pencil of rays*, whose *centre* is the *projection of the centre* of the pencil. This latter projection will be *in finity* unless the *centre lie on a* RL., the intersection of the projected plane Π and the plane Π'' through the centre S, ‖ to Π'; *then* it is *in* ∞. Hence any pencil whose centre is on this RL., IL'', is projected into a pencil of ‖ RLs. The centres of all such pencils lay on a RL. before projection; hence they lie on a RL. after projection, namely, the RL. at ∞.

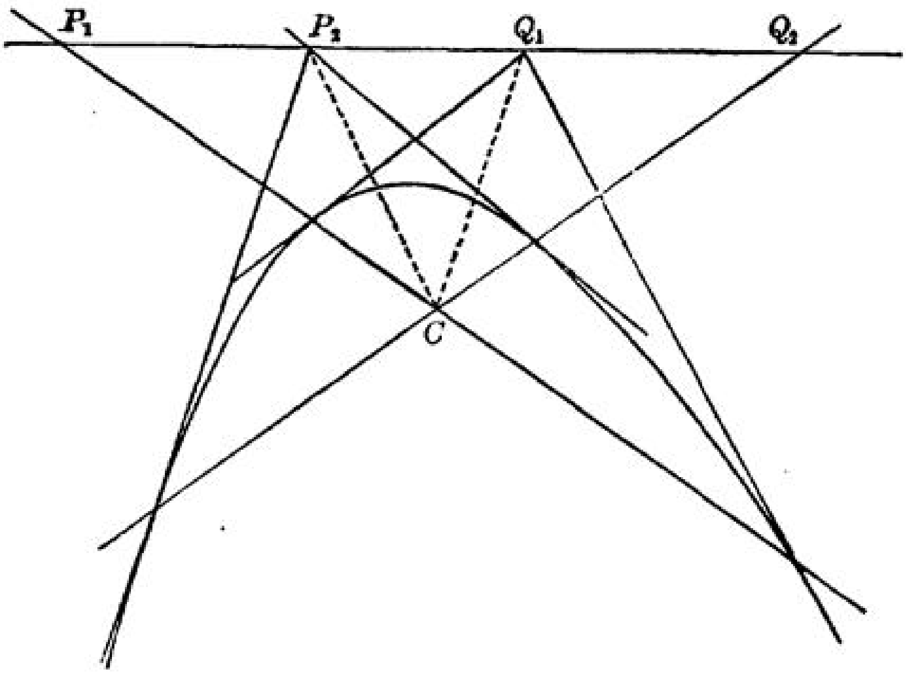
The plane Π''' through the centre S ‖ to Π meets Π in a RL. at ∞, and meets Π' in a RL. in finity, $I'''L'$; hence all *pencils* of ‖ *RLs.* in Π are projected into *pencils* of *intersecting* RLs. in Π', whose *centres lie on the RL.* $I'''L'$.

152. Since ‖ planes meet a third plane in ‖ RLs., and since $\measuredangle$s between ‖ pairs of (like-directed) RLs. are equal, clearly *any* $\measuredangle$ in Π is protected into *equal* $\measuredangle$s *on* ‖ *planes.* Hence, if the sides of any $\measuredangle$ at A cut IL'' at B and C, then the projection $\measuredangle B'A'C'$ of $\measuredangle BAC$ on Π' equals $\measuredangle BSC$, the projection of the same $\measuredangle$ on Π''. Hence, to project any $\measuredangle$ BA_1C into an $\measuredangle$ a_1, draw BS and CS making $< BSC = \measuredangle a_1$, then project on any plane ‖ to the plane BSC. This may be done in an ∞ of ways. If some RL. of the plane of BA_1C is to be projected into ∞, let it cut the sides of the $\measuredangle$ at A_1 in B_1, C_1; on B_1C_1 as chord draw in any plane an arc to contain the given $\measuredangle$ a_1; the centre S may be taken anywhere on this arc, and Π' may be any plane ‖ to the plane B_1SC_1. If *two* $\measuredangle$*s* in a plane, at A_1 and A_2, are to be projected into *two given* $\measuredangle$*s*, a_1 and a_2, while a given RL. in the plane is to be projected into ∞, construct as before in any plane on B_1C_1 as chord an arc to contain

a_1, and in the same plane on B_2C_2 as chord an arc to contain a_2: the intersection of these arcs determines the centre S, which, however, may still lie anywhere on the circle of intersection of the surfaces generated by revolving the arcs about their chords. The plane of projection Π' may be any plane ∥ to the plane through the centre S and the given RL.

On this theorem, that **any two angles** *in a plane may be projected into* **angles given** *in size and at the same time a* **given RL.** *of the plane projected into* ∞, *is based the theory of projections.* An immediate deduction is:

153. *Any* **conic** *may be projected into a* **circle** *and at the same time any* **point** *into the* **centre** *of the circle.*



Be C the point to be projected into the centre of the circle. Let the polars of any two points P_1, P_2 on the polar of C as to the conic cut that polar in Q_1, Q_2; then are P_1, Q_1 and P_2, Q_2 pairs of conjugate points. By Art. 152 project $\measuredangle P_1CQ_1$ and $\measuredangle P_2CQ_2$ each into a $R\measuredangle$ and the polar of C into ∞; then is C' the centre of the projection of the conic (which is itself a conic, by Art. 151), since its polar is at ∞; also $P_1'C'$, $Q_1'C'$ and $P_2'C'$, $Q_2'C'$ are *two* pairs of *conjugate diameters* at R$\measuredangle$s. Hence the projection is a *circle*.

If now we hold the centre of projection fast, and exchange Π and Π', we shall get the correlate theorem:

A circle may be projected into any conic, and at the same time the centre of the circle into any point of the conic.

The system of rays from S to the points of the circle form a *circular cone;* the projection of the circle on any plane is the *intersection* of that *cone* and the *plane;* hence, any *conic* is the *intersection* of a *plane* and a *circular cone*, and any *intersection* of a *plane* and a *circular cone* is a *conic.* Hence the name *conic section* or *conic.*

The student will now readily see that the section of a right circular cone is a circle when the cutting plane is $\perp$ to the axis of the cone; as the plane turns, the section passes over into an ***E*** with increasing eccentricity till the plane gets $\parallel$ to the edge of the cone, when the section becomes a ***P***; as the plane still turns, the section becomes an ***H***.

Again, if a circle stand upright on a plane, and a centre of projection descend toward the plane, the projection of the circle on the plane will be an ***E*** till the centre reaches the level of the highest point of the circle, when it becomes a ***P***; as the centre still descends, the projection becomes an ***H***; as the centre passes through the plane, the ***H*** shrinks to a pair of RLs. fallen together, and swells again into an ***E*** as the centre sinks below the plane.

Properties of a curve *not changed* by projection are called *projective properties.* By Art. 41 properties connected with the *cross-ratio* are such. But this thought cannot be followed up here.

CHAPTER VIII.

THE CONIC AS ENVELOPE.

The path here struck into leads quickly into the higher regions of the subject; we can follow it but a few steps, to find out its general direction.

154. Art. 30 has already introduced a new kind of Cds. In the Eq. $\nu_1 N_1 + \nu_2 N_2 + \nu_3 N_3 = 0$ of a RL. we may treat any two of the ratios of the three N's to each other as Cds.; or, still better, we may treat the *N's themselves* as Cds. In this case the homogeneity of the Eq. in N's shows that the apparent number of Cds., *three*, may at once be reduced to the real, *two*, by dividing by any one. In fact, if τ_1, τ_2, τ_3 be the length of the sides of the $\triangle$ of reference, it has been shown that the *triangular Cds.* of any point are connected by the relation

$$\tau_1 N_1 + \tau_2 N_2 + \tau_3 N_3 = -2A.$$

Hence, any two of such Cds. being known, the third is known. Again, any two determine a point. For all points distant N_1 from $N_1 = 0$ lie on a RL. $\parallel$ to it; so all points distant N_2 from $N_2 = 0$ lie on a RL. $\parallel$ to it; and these two RLs. meet in one, and only one, point. It is equally clear that any point determines two Cds., and therewith the third Cd. Thus it is seen that points and triangular Cds. determine each other exactly as points and Cartesian Cds.

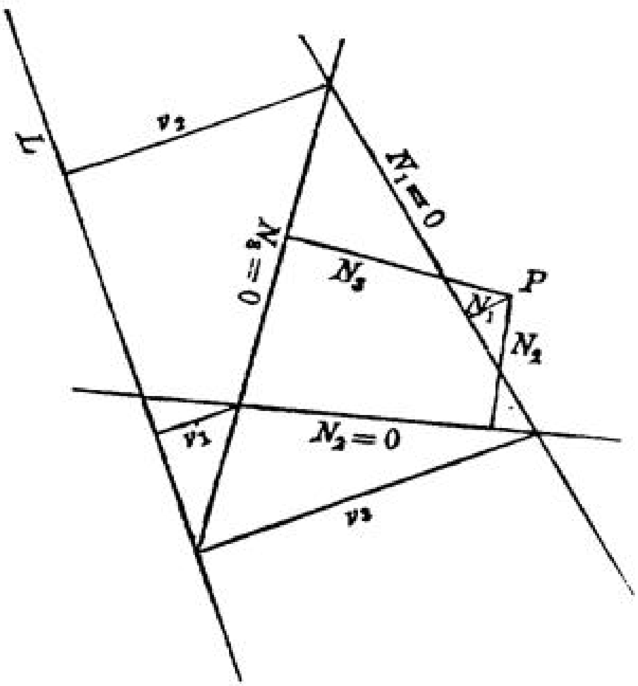
If any two ratios of the N's be taken as Cds., then are these Cds. of 0th degree in the N's, and any combination of them will still be of 0th degree; hence any Eq. between them will be homogeneous of 0th degree in N's, and on multiplication by the least common multiple of the denominators will remain homogeneous of some degree. Hence Eqs. in triangular Cds. are homogeneous.

155. Let us note more closely the Eq. of a RL. in triangular Cds.:

$$\nu_1 N_1 + \nu_2 N_2 + \nu_3 N_3 = 0. \qquad \text{(L)}$$

It is seen that the ν's enter the Eq. exactly as the N's do. The significance of this fact is now to be developed.

Holding the ν's fixed and letting the N's vary, we get various points on the same RL.; the RL. is fixed by fixing the ν's; any point on it is then fixed by fixing the N's. If, now, we hold the N's fixed and let the ν's vary, we shall clearly get various RLs. through the same point; the point is fixed by fixing the N's; any RL. through it is then fixed by fixing the ν's. Thus it seems that the ν's determine a RL. precisely as the N's determine a point.



The N's determine a point as being its distances from three fixed RLs. forming a △; how do the ν's determine a RL.? This question is easily answered thus: Take as the fixed point the vertex of the referee △ counter to the side $N_1 = 0$, i.e., the point $(N_2 = 0,\ N_3 = 0)$; then Eq. (L) reduces to $\nu_1 N_1 = 0$. Now for this point N_1 is *not* $= 0$, hence $\nu_1 = 0$; i.e., when all the RLs. go through the point $(N_2 = 0, N_3 = 0)$, then for all such RLs. $\nu_1 = 0$. Also, we know from Art. 29, that if $\nu_1 = 0$, then all the RLs. given by Eq. (L) go through the point $(N_2 = 0, N_3 = 0)$. Hence ν_1 must be a factor of, or proportional to, the distance of the RLs. from the point $(N_2 = 0, N_3 = 0)$, since when ν_1 vanishes, and only then, the

RLs. go through that point. Hence we may say $\nu_1 = 0$ is the Eq. of that point, meaning all RLs. *through it* are distant *from it* 0, just as we say $N_1 = 0$ is the Eq. of a RL., meaning all points *on it* are distant *from it* 0. Likewise $\nu_2 = 0$, $\nu_3 = 0$ are the Eqs. of the other vertices of the △, counter to the sides whose Eqs. are $N_2 = 0$, $N_3 = 0$.

As we say the point (N_2, N_3), meaning the junction-point of the RLs. $N_2 = 0$, $N_3 = 0$, so we say the RL. (ν_2, ν_3), meaning the junction-line of the points $\nu_2 = 0$, $\nu_3 = 0$.

Had we not assumed the Eq. of the RLs. in the normal form, but taken the general form $\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0$, the reasoning would have remained unchanged. Hence,

Triangular Cds. *of a point are* (fixed multiples of) *its distances from the three sides of a* △.

Triangular Cds. of a RL. are (fixed multiples of) *its distances from the three vertices of a* △.

The above interpretation of the ν's is indeed clear from Art. 30. Accordingly, Eq. (L) may be interpreted either as the *Eq. of a RL.*, the ν's being *arbitrary* and the N's *variable*, or as the *Eq. of a point*, the N's being *arbitrary* and the ν's *variable*.

The RL. fixed by the ν's chosen at will holds on it every point whose Cds. (N_1, N_2, N_3) satisfy the Eq. (L).

The point fixed by the N's chosen at will holds through it every RL. whose Cds. (ν_1, ν_2, ν_3) satisfy the Eq. (L).

The RL. is the *locus* of the *point* (N_1, N_2, N_3); the point is the *envelope* of the RL. (ν_1, ν_2, ν_3).

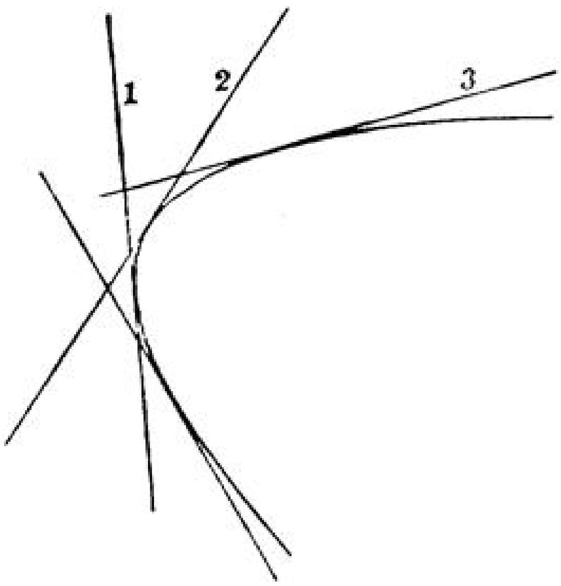
Thus far have been used two sets of symbols N's and ν's as Cds. resp. of a *point* and of a RL.; but that was plainly unnecessary, since the two sets enter Eq. (L) in the same way. We might as well say, regarding either set not as Cds., but as arbitraries, the *Eq. (L)* is the *Eq. of a RL.* or of a *point*, according as the other set be taken as the *triangular Cds.* of a *point* or of a *RL.*

156. The picture of an Eq. of higher degree between *point-Cds.* is a locus, the Cds. of each of whose points satisfies the Eq.

The picture of an Eq. of higher degree between *line-Cds.* is called an Envelope, the Cds. of each of whose RLs. satisfies the Eq.

What is meant by a point of a curve is well-known; what is meant by a RL. of a curve is to be found out.

By heaping together ever thicker and thicker points of a curve, the curve itself is not made but shadowed forth; so by heaping together RLs. of an envelope, the envelope is shadowed forth. Let 1, 2, 3 be any three such RLs. whose Cds. fulfil the Eq. of the envelope. Hold 1 fixed, and let 2 turn toward 1, its Cds. all the while fulfilling the Eq.; its section-point with 1 will move along 1, and will be definite for every position of 2; as 2 falls upon 1, *becomes coincident with* 1, it is named *consecutive* with 1, and its *section-point* with 1 is named a *point of the envelope*. The student will see at once that this reasoning is quite parallel to that in Art. 64. If now we let 3 turn toward 1, its Cds. all the while fulfilling the Eq., its section with 2, as it falls on 2, will be another *point of the envelope;* these two section-points (1, 2) and (2, 3) are plainly *consecutive points* of the envelope; hence 2, which goes through both, is a *tangent* to the envelope. Hence *a RL. of an envelope is a tangent to it;* and on this account the Cds. of the RL. are commonly called the *tangential* Cds. of the envelope or curve.



157. It is now easy to see the meaning of the *degree* of a tangential Eq. The *point-Eq.* (i.e., the Eq. in *point-Cds.*) of a RL. being of first degree, for a *point-Eq.* of a curve to be of nth degree meant there were n points common to the curve and a RL.; so, the *tangential* Eq. of a point being of first degree, for a *tangential Eq.* of an envelope to be of nth degree means there are n RLs. common to the envelope and a point; i.e., *through a point* may be drawn *n tangents* to the envelope. Such an envelope or curve is said to be of *nth class*.

Again, the condition that a *RL.* should *touch a curve* was that on combining the two point-Eqs., two sets of *point-Cds.* should fall out equal, two *common* points be *consecutive;* so, the condition that a *point* shall be *on* an *envelope* is that, on combining the Eqs., two sets of *line-Cds.* shall fall out equal, two *common* RLs. be *consecutive*.

This furnishes a general method of interchanging point-Eqs. and tangential Eqs. Combine the point-Eq. (L) of a RL. with the point-Eq. of any curve; *express the condition that two roots of the resulting Eq. be equal;* the Eq. which states this condition states that the RL. whose Eq. is (L) touches the curve; hence it is the *tangential Eq.* sought. In it the parameters (or ν's) in (L) are tangential Cds. Likewise, combine the tangential Eq. (L) of a point with the tangential Eq. of a curve; *express the condition that two roots of the resulting Eq. be equal;* the Eq. which states this condition states that the point whose Eq. is (L) is on the curve; hence it is the *point-Eq.* sought. In it the parameters (or N's) in (L) are point-Cds.

Note that the analytic work in both cases is the same.

158. For the curve of second degree a more elegant method is this:

Common Cartesian Eqs. may be made homogeneous by replacing x and y by $x:z$ and $y:z$, which plainly we may do, and then multiplying by z. The Eq. of a RL. becomes

$$lx + my + nz = 0;$$

the Eq. of a conic and the tangent to it at (x_1, y_1, z_1) become

$$kx^2 + 2\,hxy + jy^2 + 2gxz + 2fyz + cz^2 = 0,$$

$$(kx_1 + hy_1 + gz_1)x + (hx_1 + jy_1 + fz_1)y$$
$$+ (gx_1 + fy_1 + cz_1)z = 0.$$

If the RL. be a tangent, then the first and third of these Eqs. can differ only by a constant factor μ; hence

$$kx_1 + hy_1 + gz_1 - \mu l = 0,$$
$$hx_1 + jy_1 + fz_1 - \mu m = 0,$$
$$gx_1 + fy_1 + cz_1 - \mu n = 0.$$

Also, as the point (x_1, y_1, z_1) is on the tangent, the Eq. holds:

$$lx_1 + my_1 + nz_1 = 0.$$

The condition that these four Eqs. between x_1, y_1, z_1 consist, is

$$\begin{vmatrix} k & h & g & l \\ h & j & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix} = 0,$$

or $$Kl^2 + 2Hlm + Jn^2 + 2Gln + 2Fmn + Cn^2 = 0.$$

Such, then, is the *tangential Eq. of the curve of second degree.* The tangential Cds. are l, m, n, which are to be interpreted like the ν's, while the capitals are, as always, the co-factors of the like small letters in the discriminant Δ. The relation of the above determinant to Δ is to be carefully noted.

From this tangential Eq. the original point-Eq. is now to be got exactly as this tangential Eq. was got from the original point-Eq.: l, m, n will change back into x, y, z, while the capitals will change into their own co-factors in the discriminant $|KJC|$ of this Eq. The result can differ from the original Eq. by a constant factor only, by which we may divide. This factor itself is readily found thus:

Call the co-factors of K, H, etc., k', h', etc.; then

$$|KJC| = |kjc|^2, \quad \text{and} \quad |k'j'c'| = |KJC|^2;$$

$$\therefore \; |k'j'c'| = |kjc|^4.$$

But $$|\Delta k, \Delta j, \Delta c| = |kjc|^4;$$

whence $$k' = \Delta k, \text{ etc.},$$

as the student may readily verify. Hence the constant factor is the *discriminant* Δ. When, and only when, $\Delta = 1$ will $k' = k$, or the deduced be the same as the original Eq.; Δ can always be made $= 1$ by dividing the original Eq. by $\sqrt[3]{\Delta}$.

159. Since k, h, etc., are at will, their co-factors K, H, etc., are at will; hence the tangential Eq. of Art. 158 is the most general tangential Eq. of second degree; since it represents a conic, we conclude that the *general tangential Eq. of* **second** *degree represents* **a conic.** Again, since K, H, etc., are at will, so are k', h', etc.; hence the point-Eq. deduced from the tangential is the general point-Eq. of second degree; hence the general tangential Eq. represents every conic; i.e., *all curves of second degree are all curves of second class.* In general, degree and class of a curve are not of the same number.

If n points be taken on a conic and numbered consecutively from 1 to n, and each pair of consecutives be joined by a RL., the nth being joined to the first, the points will form the vertices, and the junction-lines the sides, of an inscribed n-side.

If n tangents be taken on a conic and numbered consecutively from 1 to n, and each pair of consecutives be joined by a point, the nth being joined to the first, the tangents will form the sides, and the junction-points the vertices, of a circumscribed n-side.

In this way we might now proceed to double by re-interpretation the whole body of doctrine gone before, by making proper changes in words throughout. But such detailed treatment would not be in place here. One special case of great importance may serve to illustrate.

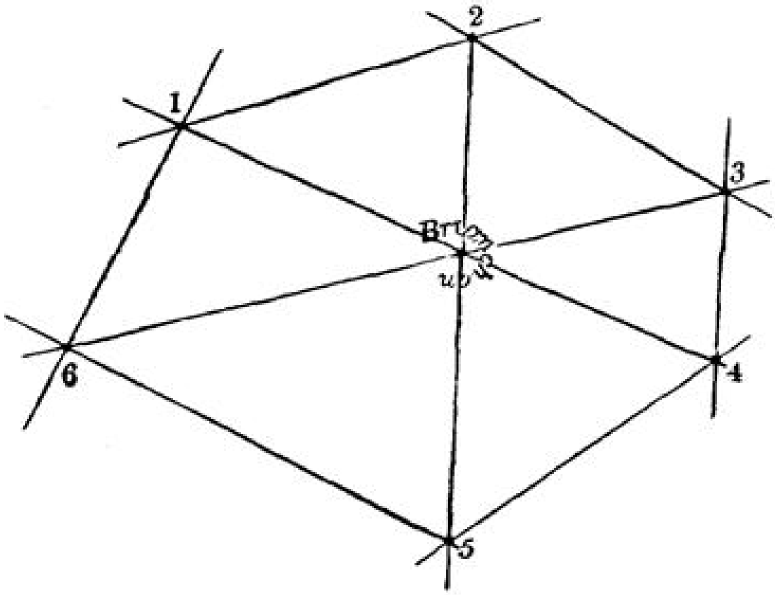
160. If $H_1 = 0$, $H_2 = 0$, $H_3 = 0$, $H_4 = 0$, $H_5 = 0$, $H_6 = 0$ are the Eqs. in *homogeneous point-Cds.* of the *sides* of a 6-side *inscribed* in a conic, they are also the Eqs. in *homogeneous line-Cds.* of the *vertices* of a 6-side *circumscribed* about a conic: the Eqs. of the two conics will be the same in form, but one will be in point-Cds., the other in line-Cds. By Pascal's Theorem the junction-points of $H_1 = 0$ and $H_4 = 0$, $H_2 = 0$

and $H_5 = 0$, $H_3 = 0$ and $H_6 = 0$ lie on a RL.; the Eq. which says this, interpreted in point-Cds., says, when interpreted in line-Cds., that the junction-lines of $H_1 = 0$ and $H_4 = 0$, $H_2 = 0$ and $H_5 = 0$, $H_3 = 0$ and $H_6 = 0$, meet in a point. Thus is found **Brianchon's** *Theorem:*

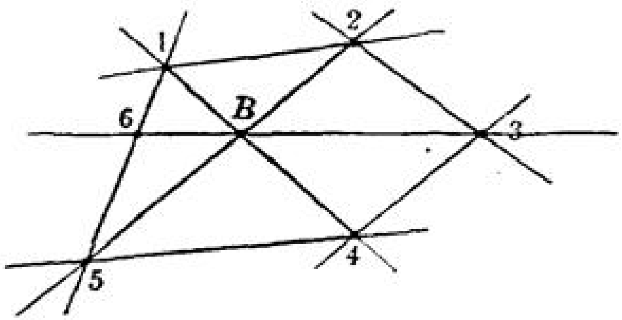
The three diagonals through the counter-vertices of a 6-side circumscribed about a conic meet in a point.

This correlate to Pascal's Theorem was first proved, though not as above, by Brianchon, a pupil of the Polytechnic School at Paris (1827). As there are sixty Pascal lines (Pascals), so there are sixty Brianchon points (Brianchons).

By Pascal's Theorem we can find a sixth point of a conic, knowing 5; by Brianchon's we can find a sixth tangent, knowing 5. For be 1, 2, 3, 4 the junction-points of the pairs of consecutive tangents, taken in order; take on the fifth tangent any point, as vertex 5; draw $\overline{1\,4}$ and $\overline{2\,5}$; through their section and 3 draw a RL.; it will cut the first side at vertex 6; then is $\overline{5\,6}$ the sixth tangent.



By Pascal's Theorem we could find the *tangent* at any *vertex;* by Brianchon's, we can find the *tangent-point* on any *tangent.* For, suppose the tangent or RL. $\overline{5\,6}$ to fall together with $\overline{6\,1}$ upon $\overline{5\,1}$; then $\overline{5\,1}$ touches at 6. Draw $\overline{1\,4}$ and $\overline{2\,5}$; through their section and 3 draw a RL., the third diagonal; it will cut $\overline{5\,1}$ at 6.



161. Line-Cds. are of special use in dealing with *loci* of *poles* and *envelopes* of *polars*. If the pole (as to any referee) move on any curve L, its polar will turn around some curve E. The junction-line of two points of L is the polar of the junction-point of the polars of those points; if these points be consecutive, their junction-line is tangent to L; then their polars are consecutive, and the junction-point of these polars is the point at which they, fallen together, touch E. Hence the *polars of all points of E are tangent to L;* i.e., as the *pole traces* L the *polar envelops* E, and as the *pole traces* E the *polar envelops* L. Hence the relation of L and E holds when the terms are exchanged; i.e., it is a *mutual* relation. L and E are called *reciprocal* curves as to the referee; this may be any conic, most simply a circle.

We may now prove Brianchon's Theorem from Pascal's thus: The poles of the six sides of the hexagon inscribed in L are the vertices of a hexagon circumscribed about E. The junction-points of pairs of counter-sides of the inscribed hexagon are poles of the diagonals (junction-lines) of counter vertices of the circumscribed; since the poles lie on a RL., the polars go through a point. Since L is any conic, so is E.

Of course, it is just as easy to prove Pascal's Theorem from Brianchon's; it is done by exchanging clauses in the sentence, " since the poles, etc." Neither theorem is logically first.

These methods of double interpretation and of exchanging the notions of pole and polar have received the names of Principles of Duality resp. Reciprocity. They are in last analysis one, and their possibility is given in the fact that the *plane*, whether viewed as *full* of *points* or *full* of *RLs.*, is *doubly extended:* there are as many points as RLs. in the plane; a point for every RL., a RL. for every point. The referee sets these points and RLs. in relation to each other.

In conclusion, it is to note in regard to reciprocal curves that, if a RL. cuts L in n points, through its pole go n RLs., polars of those points, all tangent to E; hence the degree (resp. class) of either of two reciprocal curves is of the same number as the class (resp. degree) of the other. Hence the point- (resp. tangential) Eq. of either will be of the same degree as the tangential (resp. point-) Eq. of the other. Hence the *reciprocals* of *conics* are *conics:* if the pole traces a conic, the polar envelops a conic; and conversely.

Note on Points and Right Lines at Infinity.

In view of the extensive and important use made of the notions of point and RL. at ∞, it may be well to ground these notions more thoroughly than could be done in the body of the book without breaking quite the thread of thought.

All reasoning is in first intention not about things, but about notions or concepts. It is a familiar fact of every-day life that the same thing may be conceived variously, and that the conclusions that hold about it may vary accordingly. Important illustrations have already met us. Two coincident points are in themselves one and the same point; it is only the mind that thinks the point now as on this curve, now as on that. So consecutive points are in themselves one and the same; it is only in thought that they are held apart.

The obverse of this fact is the less familiar one that things in themselves different may be, indeed, must be, thought as the same if there be no mark to distinguish them in thought. The conclusions that hold about them will be the same. Such things are all points on a RL. whose distances from any given point of that RL. are unassignably great. However apart they may be in themselves, *they cannot be held apart in thought.* Hence all such points are, for *thought, one* point, and we speak with strictest accuracy of the *one* point at ∞, the one point, not of the RL. *out of* thought, but of the RL. *in* thought.

Such, again, are all points at ∞ on parallel RLs. If $y=0$ and $y=b$ be two such RLs., then indeed the ∞ points of the two would seem to have this mark of distinction, that the y of the one is 0 while the y of the other is b. But this difference, while it distinguishes them in their outer being, does not yet distinguish them in thought. For it imparts no property to the one that does not belong to the other. By the side of the ∞ value of x, the finite value of y loses all distinguishing power. This is clearly seen on drawing a RL., say through the origin, towards the ∞ point of the RL. $y=b$; the RL. is clearly none other than the X-axis, $y=0$, for any other RL. will not extend toward the ∞ point of $y=b$, but toward some definite finite point of it. Hence we say *all parallel RLs. meet at* ∞, meaning that the marks of distinction in our notions of the points of the RLs. vanish utterly as the points are thought retiring on the RLs. without limit, leaving the notions of all the points undistinguished, one and the same. That parallel RLs. meet at ∞ is, then, no merely convenient form of speech, but states a *fact*, not of things but of thoughts.

Like reasoning applies to the RL. at ∞. It might indeed be said that there are many RLs. at ∞, for any RL. might be thought pushed to ∞ while kept in a given direction; but all such RLs. lose all distinction of direction in thought, yea, since each must go through the ∞ point of each axis, they fall together in thought completely. Hence the phrase *the RL. at* ∞ correctly expresses our thought of them.

Let the student beware of confounding the notion or concept of a thing, which is given in its definition and is the subject-matter of thought about it, with its mental image. Of points and RLs. at ∞ there are no such images at all.

EXAMPLES.*

Centre and Diameters.

1. Find the centre and the pencil of diameters of

$$5x^2+12xy-3y^2+8x-10y-7=0.$$

$$\Delta=\begin{vmatrix}5 & 6 & 4\\ 6 & -3 & -5\\ 4 & -5 & -7\end{vmatrix}=5(-4)-6(-22)+4(-18)=40,$$

$$C=-51,\quad G=-22,\quad F=-18;$$

hence the centre is $\left(\frac{22}{51},\ \frac{18}{51}\right)$; the pencil of diameters is

$$5x+6y+4+\lambda(6x-3y-5)=0;$$

the central Eq. is

$$5x^2+12xy-3y^2-\frac{40}{51}=0;$$

the curve is an ***H***.

2. Discuss in like manner these Eqs.:

$$3x^2-8xy+7y^2-4x+6y=13;$$
$$4x^2-6xy+9y^2+5x+3y=10;$$
$$7x^2-10xy+3y^2-8x-12y=2;$$

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* It has been thought best not to interrupt the development of the subject, but to put all the exercises at the close. The teacher may introduce them as he deems fit; they are arranged in the order of the foregoing text. For very many the author has to thank Hockheim's admirable collection.

$$2x^2 - 5xy - 3y^2 + 9x - 13y + 10 = 0;$$
$$3y^2 - 2xy + y^2 + 2x + 2y + 5 = 0;$$
$$13x^2 + 14xy + 5y^2 + 14x + 10y + 5 = 0.$$

3. Turn the axes so as to make the term in xy vanish.

Be $kx^2 + 2hxy + jy^2 + c = 0$ the central Eq.; put

$$x = x' \cos\alpha - y' \sin\alpha, \quad y = x' \sin\alpha + y' \cos\alpha;$$

on substitution $-2k \sin\alpha\cos\alpha + 2j \sin\alpha\cos\alpha + 2h(\overline{\cos\alpha}^2 - \overline{\sin\alpha}^2)$ must $= 0$; hence $\tan 2\alpha = \frac{2h}{k-j}$, and the Eq. takes the form

$$[k + j - \sqrt{(k-j)^2 + 4h^2}]x'^2 + [k + j + \sqrt{(k-j)^2 + 4h^2}]y'^2 + 2c = 0.$$

Illustrations: $10x^2 - 6xy + 7y^2 = 30$; $9x^2 + 16xy - 20y^2 = 60$.

4. Turn the axes till the X-axis is ∥ to the axis of the P

$$kx^2 + 2hxy + jy^2 + 2gx + 2fy + c = 0.$$

Here $C = kj - h^2 = 0$, or $h = \sqrt{kj}$; the axis of P is ∥ to

$$\sqrt{k}x + \sqrt{j}y = 0;$$

hence $\tan\alpha = -\sqrt{\frac{k}{j}} = -\frac{k}{h} = -\frac{h}{j}$; the reduced Eq. is

$$\frac{k^2 + j^2}{k} y'^2 + 2\left(\frac{hf - kg}{\sqrt{k^2 + j^2}}\right) x' + 2\left(\frac{kf + hg}{\sqrt{k^2 + j^2}}\right) y' + c = 0.$$

Illustration: $9x^2 - 6xy + y^2 + 4x + 3y + 10 = 0$.

5. A diameter of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $y = sx$; what is the conjugate?

6. Find the chord of $\frac{x^2}{11} + \frac{y^2}{13} = 1$ through $(1, 2)$ and halved by $2y = 3x$.

7. A diameter of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $y = sx$; find the conjugate.

8. Find the chord of $\frac{x^2}{36} + \frac{y^2}{9} = 1$ halved by $(4, 2)$.

The diameter through $(4, 2)$ is $2y = x$; the conjugate is $2y + x = 0$; hence the chord is $2(y - 2) + (x - 4) = 0$, or $2y + x = 8$.

9. Find the chord of $\frac{x^2}{4} - \frac{y^2}{49} = 1$ halved by $(5, 3)$.

10. In $\frac{x^2}{16} - \frac{y^2}{25} = 1$ find the ∡ between $3y = x$ and its conjugate.

11. In $25x^2 - 16y^2 = 400$ find the conjugate diameters sloped 45° to each other.

12. Find where the diameter through (2, 3) and its conjugate cut $4x^2 + 12y^2 = 48$.

13. Find lengths of the diameter $4y = 5x$ and its conjugate in $49y^2 + 9x^2 = 441$.

14. Find the slope of the diameter through (x_1, y_1) to its conjugate.

The direction-coefficients of the two diameters are $\frac{y_1}{x_1}$ and $\mp\frac{b^2}{a^2}\cdot\frac{x_1}{y_1}$;

hence $\tan\phi \dfrac{\mp\frac{b^2}{a^2}\cdot\frac{x_1}{y_1} - \frac{y_1}{x_1}}{1 \mp \frac{b^2}{a^2}} = \mp\dfrac{b^2}{e^2 x_1 y_1}$, since $a^2 y_1{}^2 \pm b^2 x_1{}^2 = \pm a^2 b^2$.

15. In $\frac{x^2}{3b^2} + \frac{y^2}{b^2} = 1$ find the conjugate diameters sloped 120° to each other.

16. Find the length of the diameter conjugate to $2x = 5y$ in $-4x^2 + 25y^2 = -100$.

17. Given $a' = 7$, $b' = 10$, $\phi = 110°$; find a and b.

Remember the relations $a'^2 + b'^2 = a^2 + b^2$, $a'b' \sin\phi = ab$.

18. Find the slopes to the axis major of the equi-conjugate diameters of $64y^2 + 25x^2 = 1600$.

19. Prove that the diagonals of the parallelogram of tangents at the ends of conjugate diameters are themselves conjugate diameters.

20. Form the axial Eq. of the conic: when $a = 13$, $ae = 12$; when $a + b = 27$, $ae = 9$; when the conic goes through (1, 4), (−6, 1); when it goes through (x_1, y_1), (x_2, y_2); when $b^2 = -144$, $ae = 13$; when $e = 3$; when $a = 4$ and the conic goes through (10, 25).

21. Find the points of a centric conic for which x and y are equal. When and how can these points be constructed?

22. Find the axes of the conic whose vertical Eq. is $y^2 = 5\frac{5}{9}x - \frac{25}{81}x^2$.

23. Find the vertical Eq. when $b = 6$ and the parameter = 5.

24. An H goes through (x_1, y_1) and the parameter is q; find the vertical Eq. and the length of the real axis.

25. Express the vertical Eq. of H through e and b as known.

26. Show from the central polar Eq. of a conic that the sum of the squared reciprocals of two ⊥ diameters is constant.

27. Find where $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$ is cut by $y = sx + c$.

28. Interpret geometrically $a^2s^2 \pm b^2 - c^2 \gtreqless 0$.

Since $s = \tan\theta$, we may write $a^2 \gtreqless a^2e^2\,\overline{\cos\theta}^2 + c^2\,\overline{\cos\theta}^2$; hence the RL. cuts the ***E*** in real, coincident, or imaginary points, according as the foot of the focal ⊥ on it lies within, upon, or without the major circle; *vice versa* for ***H***.

29. When does $kx^2 + 2hxy + jy^2 + 2gx + 2fy + c = 0$ touch either or both axes?

30. Find the Eq. of the tangents from the origin to

$$kx^2 + 2hxy + jy^2 + 2gx + 2fy + c = 0.$$

31. Find the Eq. of the RL. halving each of the positive half-axes of a centric conic; where does it cut the conic?

32. Two vertices of an equilateral hexagon inscribed in the ***E*** $25y^2 + 9x^2 = 225$ are at the ends of the axis minor; where are the others?

33. Inscribe in a centric conic a rectangle of area $2ab$.

Tangents and Normals.

34. Tangent and normal form with the X-axis an isosceles △; find its vertex, the point of tangence.

35. At corresponding points of an ***E*** and its major circle are drawn tangents; prove that the subtangents are equal. Hence frame a rule for drawing a tangent at any point of an ***E***.

36. Find the Eqs. of tangents to $5y^2 + 3x^2 = 15 \parallel$ to $3y - 4x - 1 = 0$.

37. Find Eqs. of tangents to $36y^2 + 25x^2 = 900$ sloped $30°$ to the X-axis.

38. Find the Eq. of the tangent to $9y^2 + x^2 = 9$ when the X-tangent is 5.

39. Find the point of tangence whose abscissa equals the subtangent.

40. Form the Eq. of a circle whose diameter is the tangent-intercept between the vertical tangents. Where does it cut the X-axis?

41. Form the Eq. of a circle whose centre is on the axis minor, and which has the Y-tangent as a chord. Where does it cut the other axis?

42. Find the ∡ between the tangents at (x_1, y_1) and (x_2, y_2) and the Eq. of the diameter through their intersection.

43. Find the ratio of the rectangles of the ordinates resp. abscissas of the points of touch of two ⊥ tangents.

44. Find the common tangents to a centric conic and its *mid-circle* ($x^2 + y^2 = ab$), also the ∡ under which the curves cut.

45. Find the ∡ between the tangents from (x_1, y_1) to a centric conic.

Foci and Directrices.

46. From the foot of a directrix (on the axis) is drawn a tangent to a conic; from any point of the tangent is dropped a ⊥ on the axis; from where the ⊥ cuts the curve is drawn a ray to the focus; find the ratio of the ⊥ to the focal ray.

47. Find the Cds. of the pole of $lx + my + n = 0$ as to a conic.

48. Draw a tangent to a conic at a given point, a focus and its directrix being known.

49. The diameter through $P\ (x_1, y_1)$ cuts a directrix at D; find the ∡ between the polar of P and the focal ray of D.

50. Given an ∡, the counter-side, and the sum or difference of the other sides of a △; find the sides and angles.

51. Find the sum of two focal chords ∥ to two conjugate diameters.

52. Find the rectangle of the segments of a focal chord.

53. Find the harmonic mean of the segments of a focal chord.

54. Find the harmonic mean of two ⊥ focal chords.

Asymptotes.

55. Find the ∡ between the asymptotes of $4x^2 - 5y^2 = 100$.

56. How long is the focal ⊥ on an asymptote? How long is the asymptotic intercept between the two ⊥s?

57. Find the Eq. of the tangent at (x_1, y_1) to $4xy = a^2 + b^2$.

58. When does $y = sx + c$ touch $4xy = a^2 + b^2$?

59. Find the Eq. of the RL. through (x_1, y_1), (x_2, y_2) on $4xy = a^2 + b^2$.

60. The asymptotic intercept between two chords joining two fixed points of an ***H*** to a variable point of the ***H*** is constant.

61. Find the asymptotic Cds. of the pole of the chord through (x_1, y_1), (x_2, y_2).

62. Find the asymptotic Eqs. of the directrices.

63. The asymptotic distance of a point of an *H* from a directrix equals the focal distance of the point.

64. The focal ray of a point of a conic, the $\perp$ through the focus on the ray, and the polar of the point go through a point.

The Parabola.

65. The vertex of a *P* is (a, b), the parameter is q, the axis is $\parallel$ to the X-axis; what is the Eq. of the *P*?

66. The axis of a *P* is $y = 6$, the x of the vertex is 2, and the curve goes through $(7, -8)$; find the Eq.

67. For what point of $y^2 = 4\,qx$ is y n-times x?

68. What is the Eq. of $y^2 = 10\,x$ when $\omega = 45°$, the axis being a diameter and the tangent through its end?

69. Find the rectangle of the ordinates of the ends of a focal chord.

70. Where does $y = 5\,x + c$ cut $y^2 = 4\,qx$? Interpret $q \gtreqless sc$.

71. Find the side and height of an equilateral $\triangle$ inscribed in a *P*.

72. Find the Eq. of the chord through (x_1, y_1) that is halved by (x_1, y_1).

73. Find the Eq. of the *P* whose axis is $\parallel$ to the X-axis, whose parameter is 3, which cuts the X-axis at $(12, 0)$, and touches the Y-axis.

74. Find the Eqs. of the tangents from the origin to

$$(y - b)^2 = 4\,q\,(x - a).$$

75. The vertex of a *P* is at $(a, 0)$, its axis falls on the X-axis, and it is touched by $lx + my + n = 0$; find its Eq.

76. Find the tangent-lengths and normal-lengths in *P*.

77. Find the Cds. of the point of touch of a tangent sloped ϕ to the axis.

78. When is the normal-length equal to the difference of subtangent and subnormal? When is the rectangle of the tangent- and normal-lengths equal to the square of the ordinates?

79. Find the ratio of the tracts drawn from any point of a tangent to the focus, and the foot of the $\perp$ from the point of touch on the directrix.

80. Find the Eq. of a tangent to $y^2 = 4\,qx$ sloped ϕ to $y = sx + c$.

81. Find the rectangle of the subtangents of two $\perp$ tangents.

82. Show that tangents at the ends of a focal chord are $\perp$.

83. Find the Eq. of the focal ray of the intersection of tangents at P_1, P_2.

84. An isosceles $\triangle$ is circumscribed about a **P**; show that the vertex, the point of touch of the base, and the focus lie on a RL.

85. An equilateral $\triangle$ is circumscribed about a **P**; the transversals through the vertices and the points of touch of the counter-sides go through the focus.

86. Under what $\measuredangle$s do $x^2 + y^2 = q^2$ and $\overline{x-q}^2 + y^2 = p^2$ cut $y^2 = 4qx$?

87. Find a **P** whose axis falls on the $+X$-axis, and which touches $y^2 + (x-a)^2 = r^2$ enclosing it, e.g., $y^2 + \overline{x-13}^2 = 25$.

88. Find the common tangents to $x^2 + y^2 = r^2$ and $y^2 = 4qx$.

89. Find the common tangents to the co-axial **P**'s $y^2 = 4q_1x$, $y^2 = 4q_2(x-a)$.

90. Find the $\measuredangle$ between the tangents through (x_1, y_1) to $y^2 = 4qx$.

91. What is the pole of $lx + my + n = 0$ as to $y^2 = 4qx$?

92. P is on the directrix of **P**; F is focus; the polar of P meets the curve at I, I'; PF is geometric mean of IF and FI'.

93. The diameter of a **P** through P cuts the directrix at D; show that FD is $\perp$ to the polar of P.

94. The axial intercepts of the polars of two points as to **P** equals the axial intercept of $\perp$s from them on the axis.

95. The Eq. of a pencil of $\parallel$ chords of a **P** is $y = sx + b$; what is the Eq. of the conjugate diameter?

96. If a chord cuts off equal segments from two diameters of a **P**, the diameters cut off two equal segments from the chord.

General Focal Properties.

97. The focal $\perp$ on a tangent meets the central ray to the point of touch on the directrix.

98. The focal ray to a point of a conic equals the ordinate of the point, prolonged to the tangent at an end of the focal ordinate.

99. The focal polar Eq. of a chord, the slopes of the rays to whose ends are $\theta_1+\theta_2,\ \theta_1-\theta_2$, is $2q:\rho=-e\cos\theta+\sec\theta_2\cos(\theta-\theta_1)$.

100. Hence, show that the tangent at (ρ_1, θ_1) is

$$2q:\rho=-e\cos\theta+\cos(\theta-\theta_1).$$

The Eq. of a RL. through (ρ', θ') and (ρ'', θ'') is

$$\rho\rho'\sin\overline{\theta-\theta'}+\rho'\rho''\sin\overline{\theta'-\theta''}+\rho''\rho\sin\overline{\theta''-\theta}=0.$$

Put $\theta_1+\theta_2$ resp. $\theta_1-\theta_2$ for θ' resp. θ'', and for ρ' resp. ρ'' put

$$\frac{2q}{1-e\cos\theta'}\quad\text{resp.}\quad\frac{2q}{1-e\cos\theta''},$$

since (ρ', θ'), (ρ'', θ'') are on the curve, clear of fractions; so we get, on collecting,

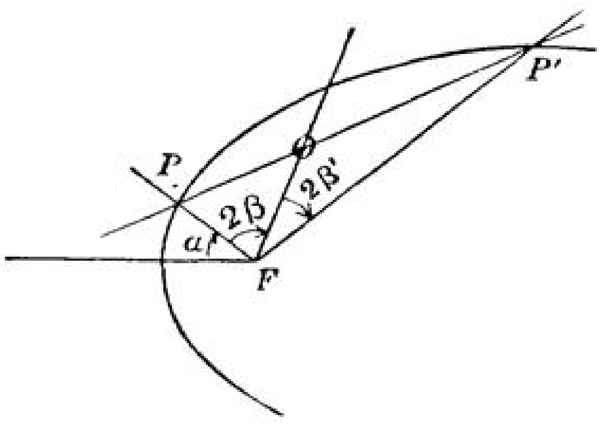
$$\rho\sin(\theta-\overline{\theta_1+\theta_2})(1-e\cos\overline{\theta_1-\theta_2})+2q\sin2\theta_2$$
$$+\rho\sin(\theta_1-\theta_2-\theta)(1-e\cos\overline{\theta_1+\theta_2})=0,$$

$$2\rho\sin(-\theta_2)\cos(\theta-\theta_1)+2q\sin2\theta_2$$
$$-e\rho\{\sin(\theta-\overline{\theta_1+\theta_2})\cos\overline{\theta_1-\theta_2}-\sin(\theta-\overline{\theta_1-\theta_2})\cos\overline{\theta_1+\theta_2}\}=0.$$

Applying the addition theorem of the sine to the bracket, we get

$$2\rho\sin(-\theta_2)\cos\overline{\theta-\theta_1}+2q\sin2\theta_2+e\rho\sin2\theta_2=0,$$

whence, on transposition and division by $\sin2\theta_2=2\sin\theta_2\cdot\cos\theta_2$, there results the Eq. of Ex. 99. In Ex. 100, $\theta_2=0$.



101. If O be fixed, PP' any chord through it, then $\tan\frac{1}{2}PFO.\tan\frac{1}{2}P'FO$ is constant.

By 99, the Eq. of PP' is

$$2q:\rho=-e\cos\theta+\sec\overline{\beta+\beta'}\cos(\theta-\overline{\alpha+\beta+\beta'}).$$

Hence, $2q : \rho_1 = -e\cos\overline{\alpha + 2\beta} + \sec\overline{\beta + \beta'}\cos(\beta - \beta')$ for the point $O(\rho_1, \alpha + 2\beta)$.

Or, $\cos\overline{\beta - \beta'} : \cos\overline{\beta + \beta'}$ is constant;

or, $\{\cos\overline{\beta - \beta'} - \cos\overline{\beta + \beta'}\} : \{\cos\overline{\beta - \beta'} + \cos\overline{\beta + \beta'}\}$ is constant;

or, $\dfrac{\sin\beta\,\sin\beta'}{\cos\beta\,\cos\beta'} = \tan\beta\,.\,\tan\beta'$ is constant.

102. Normals at the ends of a focal chord meet on the ∥ to the axis major through the mid-point of the chord.

For they meet in the centre of the circle through the ends of the chord and the other focus; the proof is now readily completed.

Hence, find the locus of the intersection of such normals.

103. Find the locus of the intersection of ⊥ normals to a ***P***.

The magic Eq. of the normal is $y - sx + 2qs - qs^3 = 0$. The ⊥ normal is $q + 2qs^2 - xs^2 - ys^3 = 0$. These Eqs. consist when, and only when,

$$\begin{vmatrix} 0 & 0 & y & x-2q & 0 & -q \\ 0 & q & x-2q & 0 & -q & 0 \\ y & x-2q & 0 & -q & 0 & 0 \\ 0 & 0 & q & 0 & 2q-x & y \\ 0 & q & 0 & 2q-x & y & 0 \\ q & 0 & 2q-x & y & 0 & 0 \end{vmatrix} = 0.$$

Multiply the first row by y, and the third by q; add to the first q times the fourth; take from the third y times the sixth; there results

$$\begin{vmatrix} 0 & q^2+y^2 & xy-2qy & 2q^2-qx \\ y & x-2q & 0 & -q \\ qx-2q^2 & xy-2qy & -(q^2+y^2) & 0 \\ q & 0 & 2q-x & y \end{vmatrix} = 0.$$

Multiply the last row by q, add to it y times the second, then add to first row $\overline{2q-x}$ times the second; there results

$$\begin{vmatrix} y(2q-x) & q^2+y^2-(2q-x)^2 & y(x-2q) \\ q(x-2q) & y(x-2q) & -(q^2+y^2) \\ y^2+q^2 & y(x-2q) & q(2q-x) \end{vmatrix} = 0.$$

Take the second row from the third, set out the factor $y^2 - qx + 3q^2$; this factor equated to 0 satisfies the Eq. of condition, making the determinant 0, and is the locus sought: a ***P*** one-fourth as large as the original, its vertex where normals at the ends of the focal chord meet. This result may be got more simply otherwise, but the above illustrates the general method and the use of determinants.

Eccentric Angle.

104. The lengths of two conjugate half-diameters, a' and b', when ϵ is the eccentric ∡ of a', are

$$a'^2 = a^2 \overline{\cos\epsilon}^2 + b^2 \overline{\sin\epsilon}^2, \qquad b'^2 = a^2 \overline{\sin\epsilon}^2 + b^2 \overline{\cos\epsilon}^2.$$

105. Find the length of a chord of an E in terms of eccentric ∡s.

$$\overline{12}^2 = a^2(\cos\epsilon_1 - \cos\epsilon_2)^2 + b^2(\sin\epsilon_1 - \sin\epsilon_2)^2,$$

or
$$\overline{12}^2 = 4\,\overline{\sin\frac{\epsilon_1-\epsilon_2}{2}}^2 \left\{ a^2 \overline{\sin\frac{\epsilon_1+\epsilon_2}{2}}^2 + b^2 \overline{\cos\frac{\epsilon_1+\epsilon_2}{2}}^2 \right\}.$$

The bracket is the squared half-diameter ∥ to the chord; call it D_3; $\overline{12} = 2D_3 \sin\frac{\epsilon_1-\epsilon_2}{2}$. Or, much more neatly, thus:

The corresponding chord of the major circle is

$$\overline{1'2'} = 2D_3' \sin\frac{\epsilon_1-\epsilon_2}{2}.$$

In projection ∥ chords are changed in the same ratio; the ∥ diameter $2a'$ or $2D'$ changes to $2D_3$, hence

$$2D_3' \sin\frac{\epsilon_1-\epsilon_2}{2} \quad \text{to} \quad 2D_3 \sin\frac{\epsilon_1-\epsilon_2}{2}.$$

106. Find the area A of the △ whose vertices are (ϵ_1), (ϵ_2), (ϵ_3).

The sides of the corresponding △′ are $2a$ multiplied by

$$\sin\frac{\epsilon_1-\epsilon_2}{2}, \quad \sin\frac{\epsilon_2-\epsilon_3}{2}, \quad \text{resp.} \quad \sin\frac{\epsilon_3-\epsilon_1}{2};$$

the double area of any triangle is the product of the sides divided by the diameter of the circumscribed circle;

$$\therefore 2\triangle' = 4aa \sin\frac{\epsilon_1-\epsilon_2}{2}\cdot\sin\frac{\epsilon_2-\epsilon_3}{2}\cdot\sin\frac{\epsilon_3-\epsilon_1}{2};$$

hence
$$\triangle = 2ab \sin\frac{\epsilon_1-\epsilon_2}{2}\cdot\sin\frac{\epsilon_2-\epsilon_3}{2}\cdot\sin\frac{\epsilon_3-\epsilon_1}{2}.$$

Clearly, also,

$$2\triangle' = aa\{\sin\overline{\epsilon_1-\epsilon_2} + \sin\overline{\epsilon_2-\epsilon_3} + \sin\overline{\epsilon_3-\epsilon_1}\};$$

hence
$$\sin\overline{\epsilon_1-\epsilon_2} + \sin\overline{\epsilon_2-\epsilon_3} + \sin\overline{\epsilon_3-\epsilon_1}$$
$$= 4\sin\frac{\epsilon_1-\epsilon_2}{2}\cdot\sin\frac{\epsilon_2-\epsilon_3}{2}\cdot\sin\frac{\epsilon_3-\epsilon_1}{2}.$$

If r be the radius of the circle about the △ $(\epsilon_1, \epsilon_2, \epsilon_3)$, then

$$2\,\Delta = \overline{12}\cdot\overline{23}\cdot\overline{31} : 2r, \quad \text{or} \quad r = \overline{12}\cdot\overline{23}\cdot\overline{31} : 4\,\Delta,$$

$$r = \frac{D_3 \cdot D_1 \cdot D_2}{ab}.$$

A focal chord c_1 is a third proportional to the axis major and the ∥ diameter; i.e., $2a : 2\,D_1 = 2\,D_1 : c_1$; hence results

$$r^2 = \frac{c_1 \cdot c_2 \cdot c_3}{16\,q}, \text{ if } 2\,q = \frac{b^2}{a}.$$

107. Find the area A of the △ of tangents touching at (ϵ_1), (ϵ_2), (ϵ_3).

The area A' of the corresponding △′ is plainly

$$A' = aa\left\{\tan\frac{\epsilon_1 - \epsilon_2}{2} + \tan\frac{\epsilon_2 - \epsilon_3}{2} + \tan\frac{\epsilon_3 - \epsilon_1}{2}\right\};$$

$$\therefore A = ab\,\{\ldots\}.$$

By applying the determinant formula for the area of a △ fixed by three RLs. we find

$$A = ab\tan\frac{\epsilon_1 - \epsilon_2}{2}\cdot\tan\frac{\epsilon_2 - \epsilon_3}{2}\cdot\tan\frac{\epsilon_3 - \epsilon_1}{2};$$

hence *the sum* and *the product* of the *three tangents* are *equal*, a relation holding only when the *sum* of the ∡s is a *multiple* of π.

108. Show that the area of the △ formed by three normals is

$$\frac{\overline{a^2 - b^2}^2}{4\,ab}\left\{\tan\frac{\epsilon_1 - \epsilon_2}{2}\cdot\tan\frac{\epsilon_2 - \epsilon_3}{2}\cdot\tan\frac{\epsilon_3 - \epsilon_1}{2}\right\}$$

$$\{\sin\overline{\epsilon_1 + \epsilon_3} + \sin\overline{\epsilon_2 + \epsilon_3} + \sin\overline{\epsilon_3 + \epsilon_1}\}.$$

Areas.

109. The side of a rhomb inscribed in an ***E*** is s, the linear eccentricity is the geometric mean of the half-axes; find the area of the ***E***.

110. In the ***E*** $25\,y^2 + 9\,x^2 = 225$ find the area of a sector whose centric ∡ reckoned from the axis major is 60°.

111. Find the ratio of the segments into which the parameter cuts an ***E***.

112. Find the ratio of the parts into which a concentric circle through the foci cuts an ***E***.

113. Find the ratio of the parts into which a confocal ***P***, with vertex at the centre, cuts an ***E***.

114. Find the segment cut off from $b^2x^2 - a^2y^2 = a^2b^2$ by $x = d$.

115. Find the area bounded by $b^2x^2 - a^2y^2 = a^2b^2$ and the RLs. $y = c$, $y = -c$.

116. Find the area bounded by an ***H***, its major circle, and $y = a$, $y = -a$.

117. An equilateral △ of side s has for its altitude the axis, for its vertex the vertex of a ***P***; find the segments cut off by its sides.

118. The segment cut off by a focal chord of a ***P*** is one-third of the trapezoid of chord, directrix, and diameters.

119. A focal ray is prolonged by twice itself; through the point thus reached is drawn a diameter to the ***P***; compare the triangular areas bounded by the ray, the curve, the diameter, and the axis.

If the ray meet the curve at (x, y), the ratio is found to be $\dfrac{28\,x + 12\,q}{x + 3\,q}$, which is 10 when the ray is ⊥ to the axis.

120. Find the ratio of the parts into which $y^2 = 4\,qx$ cuts

$$y^2 = 16\,qx - x^2.$$

121. Find the area of the figure bounded by $y^2 = 4\,qx$ and $x^2 = 4\,qy$.

122. Find the area between $y^2 = 4\,qx$ and $y^2 = 8\,q\,(x - q)$.

123. Find the area of a focal sector of a ***P***.

Determination of the Conic.

124. Determine elements of $3\,x^2 - 4\,xy + 5\,y^2 - 30\,x - 16\,y - 20 = 0$.

$\Delta = -2017$, $C = 11$, the centre $(G : C, F : C)$ is $\left(\dfrac{91}{11}, \dfrac{54}{11}\right)$, the curve is an ***E***, the central Eq. is $3\,x^2 - 4\,xy + 5\,y^2 - \dfrac{2017}{11} = 0$, the Eq. of the pair of RLs. through the centre and the intersections of the ***E*** and a concentric circle is $3\,x^2 - 4\,xy + 5\,y^2 - \dfrac{2017}{11} \cdot \dfrac{x^2 + y^2}{r^2} = 0$; these RLs. fall together in an axis when, and only when,

$$\left(3 - \frac{2017}{11\,r^2}\right)\left(5 - \frac{2017}{11\,r^2}\right) = 4,$$

or $$\frac{2017^2}{11^2\,r^4} - 8 \cdot \frac{2017}{11\,r^2} + 16 = 4 - 15 + 16 = 5\,;$$

whence $\frac{2017}{11\,r^2} = 4 \pm \sqrt{5}$; whence the axes are

$$(\sqrt{5}+1)\,x^2 + 4\,xy + (\sqrt{5}-1)\,y^2 = 0$$

and $$(\sqrt{5}-1)\,x^2 - 4\,xy + (\sqrt{5}+1)\,y^2 = 0,$$

or $$y = -\frac{\{\sqrt{5}+1\}^{\frac{1}{2}}}{\{\sqrt{5}-1\}^{\frac{1}{2}}}\,x = -\frac{\sqrt{5}+1}{2}\,x,$$

and $$y = \frac{\{\sqrt{5}-1\}^{\frac{1}{2}}}{\{\sqrt{5}+1\}^{\frac{1}{2}}}\,x = \frac{\sqrt{5}-1}{2}\,x;$$

while the half-axes are respectively

$$\frac{\sqrt{4-\sqrt{5}}\,.\,\sqrt{2017}}{11} \quad\text{and}\quad \frac{\sqrt{4+\sqrt{5}}\,.\,\sqrt{2017}}{11}.$$

125. Determine the elements of

$$x^2 - 5\,xy + y^2 + 8\,x - 20\,y + 15 = 0,$$
$$36\,x^2 + 24\,xy + 29\,y^2 - 72 + 126\,y + 81 = 0,$$
$$2\,x^2 - 5\,xy - 3\,y^2 + 9\,x - 13\,y + 10 = 0.$$

126. Determine the elements of $9\,x^2 - 6\,xy + y^2 + 4\,x + 3\,y + 10 = 0$.
$\triangle = -169:4$, $C = 0$, the curve is a $\boldsymbol{P}$; the Eq. may be written

$$(3\,x - y + \lambda)^2 = 2\,(3\,\lambda - 2)\,x - (3 + 2\,\lambda)\,y + \lambda^2 - 10,$$

or $$L^2 = L';$$

the RLs. $L = 0$ and $L' = 0$ are a diameter and a tangent at its end; they are ⊥, and ∴ are the axis and the vertical tangent when

$$3\cdot\frac{2\,(3\,\lambda-2)}{2\,\lambda+3} + 1 = 0, \quad 20\,\lambda - 9 = 0, \quad \lambda = \frac{9}{20}.$$

The parameter is

$$4\,q = \frac{\left\{4\left(3\cdot\frac{9}{20}-2\right)^2 + 4\left(-1\cdot\frac{9}{20}-\frac{3}{2}\right)^2\right\}^{\frac{1}{2}}}{3^2+1^2} = \frac{13\sqrt{10}}{100}.$$

127. Determine the elements of

$$25\,x^2 - 120\,xy + 144\,y^2 - 2\,x - 29\,y = 1,$$
$$9\,x^2 - 12\,xy + 4\,y^2 - 24\,x + 16\,y - 9 = 0,$$
$$4\,x^2 + 9\,y^2 - 8\,x + 54\,y + 85 = 0.$$

128. The linear eccentricity of $kx^2 + 2\,hxy + jy^2 + 2\,gx + 2fy + c = 0$ is $\sqrt{-\sqrt{(k-j)^2 + 4\,h^2}\,.\,\triangle : C}$.

129. **Find the conic through $(0,0)$, $(2,2)$, $(18,6)$, $(32,8)$, $(72,12)$ by** ***inspection.***

130. Find the conic through $(3,7)$, $(-2,-8)$, $(11,31)$, $(9,-2)$, $(17,1)$.

131. Find the conic through $(1,2)$, $(0,1)$, $(1:4, (\sqrt{3}+5):4)$, $(\frac{1}{2},2)$, $(7:8, (15-\sqrt{7}):8)$.

132. Find the conic through $(-8,0)$, $(3, \sqrt{11:12})$, $(4, -2:\sqrt{3})$, $(1,-1:2)$, $(6, \sqrt{7:3})$.

133. Find the ***P*** touching the axes at $(4,0)$ and $(0,3)$.

We have $K=0$, $J=0$, $C=0$, $16k+8g+c=0$, $9j+6f+c=0$, whence $(3x \pm 4y)^2 - 24(3x+4y-6)=0$; to the lower in the double sign corresponds a ***P***, to the upper a double RL. through the points of touch.

134. **Find the conic touching both axes, the Y- one at $(0,4)$ and going through $(16:3, -20:3)$, $(-3, 10+\frac{5}{2}\sqrt{15})$.**

Constructions.

135. **Given the conic drawn, to determine its elements.**

Draw a pair of pairs of ∥ chords, and through the mid-points a pair of *diameters;* these meet in the *centre.* If the conic be a ***P***, draw two chords ⊥ to the axial direction; their halver is the axis; from the foot of any ordinate lay off a subtangent double the abscissa, through its end and the end of the ordinate (on the curve) draw a RL.; it is a tangent to the curve. Through the mid-point of the tangent-tract draw a ⊥ to it; it meets the axis at the focus; the directrix is ⊥ to the axis and is counter to the focus as to the vertex.

If the conic be centric, draw on any diameter of it a half-circle, and from where this meets the conic draw to the ends of the diameter a pair of chords; they are *supplemental* and ⊥; hence, the diameters ∥ to them are *conjugate* and ⊥; i.e., are the *axes.* In the ***E***, from an end of the axis minor draw a circle with the half-axis major as radius; it cuts this axis in the foci. But, in the ***H***, the ends of the (minor or) conjugate axis being imaginary, draw the vertical tangents (⊥ to the real axis); draw the diameter of any two ∥ chords, and at its end a tangent (∥ to the chords); on the intercept of this tangent between the ∥ tangents as diameter, draw a circle; it cuts the real axis in the *foci.* A circle about the centre and through the foci meets the vertical tangents on the *asymptotes.* Combining the Eqs. of an asymptote and the major circle, we see they meet on the *directrices;* but this construction of the latter is available only in case the asymptotes are real, i.e., in ***H***. In the ***E***, draw a tangent at the end of a focal ordinate; it cuts the axis major on the *directrix* corresponding to that focus.

136. To draw a tangent at a point to a conic.

In ***P***, lay off from the foot of the ordinate the subtangent double the abscissa; thus is reached a second point of the tangent. In ***E***, draw a tangent at the *corresponding* point to the major circle; it cuts the major axis in a second point of the tangent to ***E***. In ***H***, halve the inner ∡ of the focal rays to the point. The like construction of course holds for ***P*** and ***E***.

137. To draw a tangent from a point to a conic.

On a focal tract to the point, as diameter, draw a circle; it meets the major circle in two second points of the two tangents. In ***P*** the vertical tangent is the major circle.

138. Given the foci and one point (or tangent); construct the conic.

To give the focus at ∞ is the same as to give the direction of the axis of ***P***.

139. Given a focus, an axial direction, a tangent and its point of touch.

Use the *counter-circle* of the other focus.

140. Given $2a$, a focus, a tangent and its point of touch (or an asymptote).

141. Given a focus, two tangents (and their points of touch in ***E*** and ***H***).

142. Given a focus, and one diameter in length and position.

Find other focus and the axes, and use the major circle.

143. Given three tangents and a focus.

Use major and counter circles.

144. Given a focus, two tangents, and the axial directions.

145. Given the centre, a focus, and a tangent; find where tangents from a point P touch the conic.

On the focal tract FP as diameter draw a circle; through its intersections with the major circle draw RLs. from P; they are the tangents; to find points of touch, use the counter-circle.

146. Given the centre, axial directions, a tangent and its point of touch P.

Through P and the intersection of tangent and axis minor draw a circle with centre on the axis minor; it passes through the foci.

147. Given the centre, a tangent, and $2a$. Use the major circle.

148. Given the centre O, a point P of an ***E***, and $2a$.

Find the corresponding point P'; draw through P a ∥ to axis major cutting OP'' at B; OB is axis minor in length.

149. Given the centre, axial directions, and two points of the conic, P_1, P_2.

Express a^2 and b^2 through the Cds. of P_1 and P_2.

150. Given two tangents and their points of touch, and the direction of the axis major.

Construct a diameter and apply 149.

151. Given a point P of an ***E*** and the axis minor in length and position.

Draw ∥ to axis major, through P, cutting minor circle at P''; the central ray through P'' cuts the ordinate of P on the major circle.

152. Given a tangent and the axis major in length and position.

153. Given two conjugate diameters, AA', BB', in length and position.

Draw a tangent at B; lay off a' on the normal from B; through the end of a' and the centre draw a circle with centre on the tangent; it cuts the tangent on the axes.

154. Given any pole P and its polar L, a directrix, and the position of the axis major.

Let L meet the directrix at D; on PD as diameter, draw a circle · it cuts the axis major at a focus.

155. Given the asymptotes and the foci, or the transverse axis.

156. Given the transverse axis and a point of the ***H***, $b^2 = \dfrac{a^2y^2}{x^2 - a^2}$.

157. Given the asymptotes and a point P of the ***H***.

Draw through P tracts ending in the asymptotes; from P lay off on the longer segment the difference of the two segments; so are got any number of points of the ***H***. Draw the tangent at P ∥ to the fourth harmonic to the asymptotes and the diameter through P; the asymptotic Cds. of the vertex are each the geometric mean of the asymptotic Cds. of P.

158. Given the asymptotes and a tangent.

Halve the tangent intercept.

159. Given the asymptotes and the difference of a and b.

160. Given an asymptote, a tangent, its point of touch, and a second point.

161. Given the centre, an asymptote, a tangent, and the ratio $a : b$.

162. Given the centre, an asymptote, and two points.

163. Given an asymptote and three points.

164. Given the vertical tangents, point of touch of one, and a third tangent.

165. Given two tangents and the focus of a ***P***; find the point whose focal ray is the half-sum of the focal rays to the points of touch.

166. Given focus and (1) two points, or (2) one point and a tangent of ***P***.

In (1) draw about each point a circle through the focus; either outer common tangent is directrix; in (2) either tangent to the one focal circle from the counter-point of the focus as to the given tangent is *directrix*.

167. Given the directrix and two points (or one point and axis) of a ***P***.

168. Given the directrix, a tangent, and a point of a ***P***.

169. Given the vertex, the axis, and a point of a ***P***.

170. Given the axis, a tangent, and its point of touch (or the vertex).

171. Given the vertical tangent, another tangent, and its point of touch.

172. Given two tangents and their points of touch, in a ***P***.

173. Given the vertical tangent and two others, in a ***P***.

174. Given three tangents (or two points) and the axis of a ***P***.

175. Given four tangents to a ***P***. Use focal circles.

176. Sides, altitude, base of an isosceles △ are tangents, axis, chord of a ***P***.

177. A ***P*** touches one side of a △ at its mid-point, and the others prolonged.

178. Given the axis, the parameter, and a point of a ***P***.

Draw normal first.

179. Given the directrix (or focus, or axis), a pole, and its polar as to a ***P***.

Loci.

180. Find the locus of a point of a tract whose ends move on fixed RLs.

181. Fixed are the base of a △ and the point of touch of escribed circle; find locus of vertex.

182. The product of ⊥s from P on two RLs. is k^2; where is P?

183. Given a side and counter ∡ of a △; find locus of its mass-centre.

184. Find the locus of the fifth noteworthy point in a △, given a pair of counterparts, or a pair of adjacent parts.

185. Find the locus of mid-points of chords of an ***E***, through a point.

186. From two points on a tangent to a circle, d apart, are drawn two other tangents; where do they meet?

187. An ***E*** turns about its centre; where it cuts a fixed RL. tangents are drawn to it; where do they meet?

188. A circle intercepts given lengths on two given RLs.; where is its centre?

189. Find the locus of mass-centre of a △ of constant area, two of whose sides are fixed.

190. A △ is formed by a fixed RL. and two sides of a given ∡ turning about a fixed point; find the locus of centre of circumscribed circle.

191. Tangents from P to ***P*** form with the polar of P a △ of constant area; find the locus of P.

192. A vertex of a △ is fixed, the constant counter-side is pushed along a RL.; find the locus of the centre of the circumscribed circle.

193. The base of a △ is given; the vertex glides on $y + nx^2 = mx$, whose directrix is ∥ to the base; find the locus of centre of mass of the △.

194. Find the locus of pole of tangent to $y^2 = 4qx$ as to $x^2 + y^2 = r^2$.

Homogeneous Co-ordinates.

195. The Eq. of $x \cos \alpha + y \sin \alpha - p = 0$ in homogeneous Cds. (N_1, N_2, N_3) is $\nu_1 N_1 + \nu_2 N_2 + \nu_3 N_3 = 0$; find ν_1, ν_2, ν_3.

We have $N_1 = x \cos \alpha_1 + y \sin \alpha_1 - p_1 = 0$, and so for N_2, N_3; hence,

$$\nu_1 \cos \alpha_1 + \nu_2 \cos \alpha_2 + \nu_3 \cos \alpha_3 = \cos \alpha,$$

$$\nu_1 \sin \alpha_1 + \nu_2 \sin \alpha_2 + \nu_3 \sin \alpha_3 = \sin \alpha,$$

$$\nu_1 p_1 + \nu_2 p_2 + \nu_3 p_3 = p;$$

$$\nu_1 = |\cos \alpha \sin \alpha_2 p_3| : \Delta,$$

$$\nu_2 = |\cos \alpha_1 \sin \alpha p_3| : \Delta,$$

$$\nu_3 = |\cos \alpha_1 \sin \alpha_2 p| : \Delta,$$

$$\Delta = |\cos \alpha_1 \sin \alpha_2 p_3|.$$

196. Base lines are $L_1 = 5x - 2y + 1 = 0$, $L_2 = 2x - y - 3 = 0$, $L_3 = x + y + 1 = 0$; find the Eq. of $x + 2y - 4 = 0$ in homogeneous Cds. (L_1, L_2, L_3).

We have $5\lambda_1 + 2\lambda_2 + \lambda_3 = 1$, $-2\lambda_1 - \lambda_2 + \lambda_3 = 2$, $\lambda_1 - 3\lambda_2 + \lambda_3 = -4$; whence, on finding $\lambda_1, \lambda_2, \lambda_3$, substituting in $\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0$, and multiplying by 23, $-20L_1 + 39L_2 + 45L_3 = 0$.

197. Show that the two Eqs.

$$\tau_1 N_1 + \tau_2 N_2 + \tau_3 N_3 = 0 \quad \text{and} \quad N_1 \sin A_1 + N_2 \sin A_2 + N_3 \sin A_3 = 0,$$

represent each the RL. at ∞.

198. Where does $\nu_1 N_1 + \nu_2 N_2 + \nu_3 N_3 = 0$ cut the sides of the referee $\triangle$?

Find the intersection with $N_1 = 0$ from $\nu_2 N_2 + \nu_3 N_3 = 0$, $\tau_2 N_2 + \tau_3 N_3 = 2\Delta$, whence $N_2 = 2\Delta\nu_3 : (\tau_2\nu_3 - \tau_3\nu_2)$, $N_3 = 2\Delta\nu_2 : (\tau_3\nu_2 - \tau_2\nu_3)$, and so for the others.

199. Where do $\nu_1 N_1 + \nu_2 N_2 + \nu_3 N_3 = 0$ and $\nu_1' N_1 + \nu_2' N_2 + \nu_3' N_3 = 0$ meet?

Since $\tau_1 N_1 + \tau_2 N_2 + \tau_3 N_3 = 2\Delta$, $N_1 = 2\Delta|\tau_2\nu_3'| : |\nu_1\nu_2'\tau_3|$, and so on.

200. Find the Eq. of the RL. through (N_1', N_2', N_3') and (N_1'', N_2'', N_3'').

Assume $\nu_1 N_1 + \nu_2 N_2 + \nu_3 N_3 = 0$; then $\nu_1' N_1' + \nu_2 N_2' + \nu_3 N_3' = 0$, $\nu_1 N_1'' + \nu_2 N_2'' + \nu_3 N_3'' = 0$; $\therefore |N_1 N_2' N_3''| = 0$.

201. Find the RLs. through (N_1', N_2', N_3') and the vertices of the referee $\triangle$.

202. The RLs. $N_1 = 0$, $N_2 = 0$, $\nu_2 N_2 + \nu_3 N_3 = 0$, $\nu_1' N_1 + \nu_3' N_3 = 0$ form a four-side; find the diagonals and where they meet.

203. Show that the diagonals of a four-side are cut harmonically.

204. Homogeneous Eqs. of $\parallel$ RLs. differ only by constants.

205. Find Eqs. of RLs. through the vertices of the referee $\parallel$ to the counter-sides.

206. Find Eqs. of RLs. through the vertices and the mass-centre of the referee.

207. When are $\nu_1 N_1 + \nu_2 N_2 + \nu_3 N_3 = 0$ and $\nu_1' N_1 + \nu_2' N_2 + \nu_3' N_3 = 0$ perpendicular?

208. Find Eqs. of the altitudes of the referee and the Cds. of the orthocentre.

209. Find Eqs. of junction-lines of the feet of the altitudes of the referee.

210. Find Eqs. of the mid-perpendiculars to the sides of the referee.

211. Find Eqs. of RLs. through the vertices of the referee and the points of touch of the inscribed and escribed circles.

212. Find the distance of (N_1', N_2', N_3') from $\nu_1 N_1 + \nu_2 N_2 + \nu_3 N_3 = 0$.

When but two Cds. are used to fix a RL., call them l and m, and write the Eq. of the enwrapt point thus:

$$P = ul + vm + 1 = 0.$$

213. Find the Cds. of the junction-line of $P_1 = 0$ and $P_2 = 0$.

Proceeding exactly as if to find the junction-point of $L_1 = 0$, $L_2 = 0$, we find $l = (v_1 - v_2) : |u_1 v_2|$, $m = -(u_1 - u_2) : |u_1 v_2|$.

214. Find the distance of $P_1 = 0$ from the RL. (l_1, m_1).

To say the Cds. of the RL. are l_1, m_1, is to say its Eq. in rectilinear Cds. u, v, is $l_1 u + m_1 v + 1 = 0$; to say the Eq. of the point is $u_1 l + v_1 m + 1 = 0$, is to say its rectilinear Cds. are u_1, v_1, since they fulfil the Eq. of any RL. through it; hence, the distance sought is

$$\frac{u_1 l_1 + v_1 m_1 + 1}{\sqrt{l_1^2 + m_1^2}}.$$

215. Find the Eq. of the point that cuts the tract between $P_1 = 0$, and $P_2 = 0$ in the ratio $n_1 : n_2$.

Think of (u_1, v_1) (u_2, v_2) as the points in rectilinear Cds., then

$$\left(\frac{n_1 u_2 + n_2 u_1}{n_1 + n_2}, \frac{n_1 v_2 + n_2 v_1}{n_1 + n_2}\right)$$

is the cutting point; the Eq. of this point, viewed as enwrapped by the varying RL. (l, m), is

$$\frac{n_1 u_2 + n_2 u_1}{n_1 + n_2}\, l + \frac{n_1 v_2 + n_2 v_1}{n_1 + n_2}\, m + 1 = 0.$$

For n_1 and n_2 like-signed, the point is an inner one, otherwise an outer one. The Eq. may also be written

$$\frac{n_1 P_2 + n_2 P_1}{n_1 + n_2} = 0.$$

The *inner* resp. *outer* mid-point is $P_1 + P_2 = 0$ resp. $P_1 - P_2 = 0$.

216. The vertices of a $\triangle$ are $P_1 = 0$, $P_2 = 0$, $P_3 = 0$; find the mass-centre. From (215) it is seen to be $(P_1 + P_2 + P_3) : 3 = 0$.

217. Three vertices of a parallelogram are $P_1 = 0$, $P_2 = 0$, $P_3 = 0$; find the fourth.

218. Find Eq. of any point on junction-line of $P_1 = 0$ and $P_2 = 0$. It is $P_1 - kP_2 = 0$; for this is Eq. of a point, being of first degree in

l, m, and it lies on a RL. through $P_1 = 0$ and $P_2 = 0$, since it is fulfilled where they are. From (215) it is seen that k is the ratio of the distances of the point from $P_1 = 0$ and $P_2 = 0$; varying k gives all points on the RL.

219. When do $P_1 = 0$, $P_2 = 0$, $P_3 = 0$ lie on a RL.?

Clearly when $|u_1 v_2 1| = 0$, or when $k_1P_1 + k_2P_2 + k_3P_3 = 0$.

220. Show that the points at ∞ on the sides of a $\triangle$ lie on a RL.

221. The product of the ratios in which the sides of a $\triangle$ are cut is 1; then, and only then, the cutting points are on a RL.

The cutting points are $P_1 - k_3P_2 = 0$, $P_2 - k_1P_3 = 0$, $P_3 - k_2P_1 = 0$; the determinant of these three Eqs. vanishes only when $k_1 . k_2 . k_3 = 1$.

222. How do $P_1 + kP_2 = 0$ and $P_2 + kP_1 = 0$ lie on the RL. (P_1, P_2)?

223. The cross-ratio of $P_1 - k_nP_2 = 0$, $k = 1, 2, 3, 4$, is

$$\overline{k_1 - k_2} . \overline{k_3 - k_4} : \overline{k_2 - k_3} . \overline{k_4 - k_1}.$$

All the problems of the text as to rays may be repeated as to points. All the problems in homogeneous point-Cds. may now be paralleled by problems in homogeneous line-Cds. *E.g.:*

224. Be $P_1 = 0$, $P_2 = 0$, $P_3 = 0$ three vertices of a $\triangle$; express $P = 0$ through them in the form $\kappa_1P_1 + \kappa_2P_2 + \kappa_3P_3 = 0$. See (195).

225. Writing $p = P : \sqrt{l^2 + m^2}$, show that in $\kappa_1 p_1 + \kappa_2 p_2 + \kappa_3 p_3 = 0$, the κ's are proportional to the distances of this point from the sides of the referee.

226. Find Cds. of the RL. through $\kappa_1 p_1 + \kappa_2 p_2 + \kappa_3 p_3 = 0$ and $\kappa'_1 p_1 + \kappa'_2 p_2 + \kappa'_3 p_3 = 0$.

227. Find Eq. of junction-point of RLs. (p'_1, p'_2, p'_3) and (p''_1, p''_2, p''_3).

228. When is $\kappa_1 p_1 + \kappa_2 p_2 + \kappa_3 p_3 = 0$ a point at ∞?

The RLs. (p_1, p_2, p_3) and $(p_1 + d, p_2 + d, p_3 + d)$ are $\parallel$; since they go through the same point at ∞, their Cds. satisfy the same Eq.; or,

$\kappa_1 p_1 + \kappa_2 p_2 + \kappa_3 p_3 = 0$ and $\kappa_1(p_1 + d) + \kappa_2(p_2 + d) + \kappa_3(p_3 + d) = 0$;

whence $$\kappa_1 + \kappa_2 + \kappa_3 = 0.$$

229. Show that the mass-centre of the $\triangle$ $p_1 = 0$, $p_2 = 0$, $p_3 = 0$ is $p_1 + p_2 + p_3 = 0$.

230. The centre of the circle about the referee is

$$p_1 \sin 2A_1 + p_2 \sin 2A_2 + p_3 \sin 2A_3 = 0.$$

231. The orthocentre of the referee is

$$p_1 \tan A_1 + p_2 \tan A_2 + p_3 \tan A_3 = 0.$$

232. Mass-centre, orthocentre, and centre of vertices of a $\triangle$ lie on a RL. In the determinant of the Eqs. of the points multiply first row by $2 \sin A_1 \sin A_2 \sin A_3$; take from third row; take out $2 \cos A_1 \cos A_2 \cos A_3$.

233. The centre of the circle in the referee is

$$p_1 \sin A_1 + p_2 \sin A_2 + p_3 \sin A_3 = 0.$$

234. The intersection of the transversals from the vertices to the points of touch of the escribed circles is

$$p_1 \cot \frac{A_1}{2} + p_2 \cot \frac{A_2}{2} + p_3 \cot \frac{A_3}{2} = 0.$$

235. The points of (229), (233), (234) lie on a RL.

236. The point $\kappa_1 p_1 + \kappa_2 p_2 + \kappa_3 p_3 = 0$ is distant from the RL. (p'_1, p'_2, p'_3) $(\kappa_1 p'_1 + \kappa_2 p'_2 + \kappa_3 p'_3) : (\kappa_1 + \kappa_2 + \kappa_3)$.

Envelopes.

237. When does the RL. (l, m) touch the ***P*** whose axis falls on the $+X$-axis and its focus on the point $ql + 1 = 0$?

The RL. $lu + mv + 1 = 0$ touches $y^2 = 4qu$ when the roots of $v^2 . \frac{l}{4q} + mv + 1 = 0$ are equal, or when $m^2 4q = l$, which is therefore the Eq. of the ***P*** in line-Cds.

238. Through a fixed point P is drawn a RL., to which, where it meets a fixed RL., is drawn a $\perp$; find the envelope of the $\perp$.

Take the fixed RL. as Y-axis, the $\perp$ to it through P as X-axis; be u and v the intercepts of the enveloping RL.; then $v^2 = pu$, i.e., *the envelope is a **P**, P* is the focus.

239. Find the tangential Eq. of ***E*** referred to its axes. The intercepts are $u = a^2 : x_1$, $v = b^2 : y_1$; whence, on squaring, inverting, and putting $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$, results $\frac{a^2}{u^2} + \frac{b^2}{v^2} = 1$; or, calling the reciprocals of the intercepts l and m, $a^2 l^2 + b^2 m^2 = 1$. Otherwise, thus:

$$\Delta = \begin{vmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & -1 \end{vmatrix}; \text{ whence } K = -\frac{1}{b^2}, \; J = -\frac{1}{a^2}, \; C = \frac{1}{a^2 b^2};$$

whence, on substituting and clearing, the same result is got.

240. About the point (e, o) is drawn a circle with radius $2a$, from $(-\epsilon, o)$ is drawn a ray to the circle; find the envelope of its *mid-perpendicular* $(a > e)$.

The Eqs. of the circle and ray are $\overline{x-e}^2 + y^2 = 4a^2$, $y = s(x+e)$; that of the mid-perpendicular is

$$\left(x - \frac{\sqrt{a^2(1+s^2) - e^2s^2} - es^2}{1+s^2}\right) + s\left(y - \frac{s\sqrt{a^2(1+s^2) - e^2s^2} + se}{1+s^2}\right) = 0.$$

The intercepts of this are

$$\sqrt{a^2(1+s^2) - e^2s^2} = u \quad \text{and} \quad \sqrt{a^2(1+s^2) - e^2s^2} : s = v;$$

whence eliminating s and putting l, m for $1:u$, $1:v$, we get

$$a^2l^2 + (a^2 - e^2)m^2 = 1:$$

the envelope is an $\boldsymbol{E}$.

241. Through (e, o) and $(-e, o)$ in the circle $x^2 + y^2 = a^2$ are drawn ∥ chords; find the envelope of the RL. joining two ends of the chords on the same side of a diameter. $a^2l^2 + (a^2 - e^2)m^2 = 1.$

242. The ∥s $y = a$, $y = -a$, meet the circle $x^2 + y^2 - 2\lambda x = e^2$, and the points of meeting are joined crosswise; find envelope of junction-line when λ varies. $a^2m^2 + (a^2 - e^2)l^2 = 1.$

How does the envelope change as e changes?

243. The pole, as to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, traces the circle $x^2 + y^2 = a^2$; what does the polar envelop? *Ans.* $a^4l^2 + b^4m^2 = a^2$.

HINT. Eq. of the polar is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$; here $\frac{x_1}{a^2} = l$, $\frac{y_1}{b^2} = m$.

244. Find the envelopes when the pole traces $x^2 + y^2 = b^2$ and $a^2 + b^2$.

245. Find the envelope of the junction-line of the ends of two conjugate diameters. *Ans.* The $\boldsymbol{E}$ $a^2l^2 + b^2m^2 = 2$.

HINT. The ends are the points $(a\cos\epsilon, b\sin\epsilon)$ and $(-a\sin\epsilon, b\cos\epsilon)$; the RL. through them is $xb(\sin\epsilon - \cos\epsilon) - ya(\sin\epsilon + \cos\epsilon) + ab = 0$; here $(\cos\epsilon - \sin\epsilon) = al$, $(\cos\epsilon + \sin\epsilon) = bm$; hence the above result, on squaring.

246. From the point (e, o) rays are drawn to the circle

$$(x+e)^2 + y^2 = a^2;$$

find the envelope of their mid-perpendiculars.

247. The vertex of a right ∡ glides on $x^2 + y^2 = r^2$, one side enwraps the point (e, o); what does the other side enwrap?

Ans. $r^2l^2 - (e^2 - r^2)m^2 = 1.$

248. What is the tangential Eq. of $7x^2 - 5y^2 + 12x + 8y - 47 = 0$?

249. Show, in two ways, that the tangential Eq. of ***H*** referred to its asymptotes is $lm(a^2 + b^2) = 1$.

250. When does the general tangential Eq. of second degree,

$$Kl^2 + 2Hlm + Jm^2 + 2Gl + 2Fm + C = 0,$$

represent an ***E***? when an ***H***? when a ***P***?

Be $kx^2 + 2hxy + jy^2 + 2gx + 2fy + c = 0$ the Cartesian Eq. from which the tangential is got by putting for k, h, etc., their co-factors K, H, etc., in Δ; also suppose $\Delta > 0$. Then, as both Eqs. picture the same curves, the criterion is the same for both: the curve is ***E***, ***P***, ***H***, according as $C > 0$, $C = 0$, $C < 0$.

251. Putting k', h', etc., for the co-factors of K, H, etc., in Δ, show that the tangential Eq. pictures two points when $\Delta = 0$ and $c' < 0$, pictures one double point when $c' = 0$, $h' = 0$, $k' = 0$.

252. Discuss 250 and 251 geometrically, remembering that from *every* point *of the RL. at* ∞ may be drawn two tangents to ***E***; only from *outer* points may they be drawn to ***H***; from *every* point may be drawn only one tangent to ***P***, since the *RL. at* ∞ itself touches ***P***; and combine with the given Eq. of second degree the Eq. $l - \lambda m = 0$ of a point at ∞.

253. From a point on the X-axis are dropped $\perp$s on the RLs. $x = y$ and $x + 2y = 10$; find the envelope of the junction-line of the feet of the $\perp$s. *Ans.* Λ ***P***.

254. Two RLs. mutually $\perp$ turn about a fixed point; find the envelope of the junction-line of their intersections with two fixed RLs.

255. Through (o, d) is drawn the secant $x + y = d$ of the system of circles $x^2 + y^2 - 2\lambda x = d^2$; find the envelope of the tangents at the points of secancy.

256. Through (o, d) are drawn secants to $x^2 + y^2 = r^2$, and $\perp$s drawn to the secants at the points of secancy; find their envelope.

257. Find the envelope of the polars of a point as to a system of confocal conics.

258. A secant cuts a system of confocals; find the envelope of tangents at the points of secancy.

259. The two points at ∞ in: ***H*** are real; ***P***, coincident; ***E***, imaginary.

260. Two $\triangle$s, one formed by tangents to a curve, the other by chords joining the points of tangence, may be called *outer* resp. *inner*, and said to *correspond*. Show that in ***P*** the *outer* is half the *inner*.

261. Establish *Carnot's Theorem:* The product of all the ratios in which a conic cuts the sides of a closed polygon is 1.

HINT. By Art. 72 the product of the ratios in which the conic $F = 0$ cuts the side P_1P_2 is $F_1 : F_2$.

262. If $F_1 = k_1x^2 + 2h_1xy + j_1y^2 + 2g_1x + 2f_1y + c_1$, and λ be a parameter, then $F_1 + \lambda F_2 = 0$ is a system of conics: through how many points? what pairs of RLs. belong to the system? what ***P***'s? when does a circle belong to it? what is the locus of the centres? how lie the polars of a point as to the members of the system? how lie the poles of a given RL.?

263. Show that any RL. cuts the system in a system of points in *Involution*, whose *foci* are the points cutting harmonically the chords of the base conics.

264. Show that when the pole traces a RL. the perpolar envelops a ***P***.

265. Find the envelope of normals to an ***E***, an ***H***, a ***P***.

266. Find the envelope of the perpolar when the pole traces a conic.

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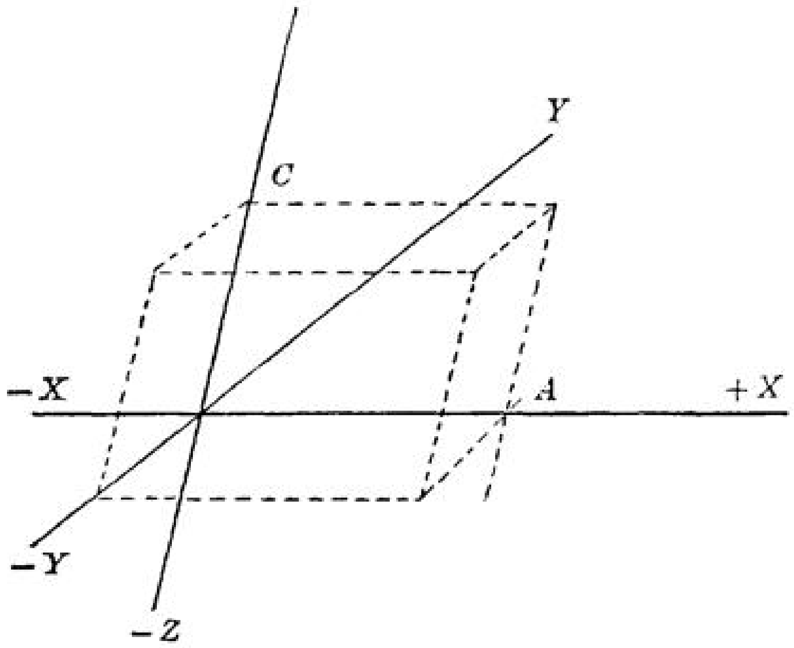
PART II. OF SPACE.

THIS subject is much more extensive than the *Geometry of the Plane*, so that any detailed treatment here is out of the question; only the most essential notions can be developed At the same time, the close analogy of the two doctrines permits a much more condensed discussion than was possible in Part I.

CHAPTER I.

1. We say of Space, it is *triply* extended or has *three* dimensions, meaning that *three* determinations are needful and enough to fix any element of it. These determinations may be made in many ways, giving rise to as many systems of *determining*



magnitudes, **Co-ordinates.** Thus, suppose three planes meeting at ***O***; call their intersections the ***X***-, ***Y***-, ***Z***- *axes*, the planes themselves the ***YZ*-, *ZX*-, *XY*-** *planes:* when no confusion would arise, omit the words *axes* and *planes*. A plane ‖ to ***YZ***, cutting off a tract $\boldsymbol{OA = a}$ on ***X***, has *all* its points at a distance a from ***YZ*** measured ‖ to ***X***, and *no other* points are so distant; hence it is defined completely by the Eq. $\boldsymbol{x = a}$. So, too, $\boldsymbol{y = b}$, $\boldsymbol{z = c}$, are *Eqs. of planes* ‖ to ZX, XY resp. The first pair meet in a RL. ‖ to Z, for *all* points of which, and for *no others*, the relations hold: $\boldsymbol{x = a, y = b}$, which are therefore the *Eqs. of a RL.* ‖ to Z.

So, too, $y = b$, $z = c$ resp. $z = c$, $x = a$ are *Eqs. of RLs.* ‖ to X resp. Y. As a special case, $x = 0$, $y = 0$, $z = 0$ are the Eqs. of YZ, ZX, XY; $y = 0$, $z = 0$, and $z = 0$, $x = 0$, and $x = 0$, $y = 0$ are the Eqs. of X and Y and Z. The three planes and three RLs. of intersection meet in a point for which, and which alone, hold all three relations: $\boldsymbol{x = a}$, $\boldsymbol{y = b}$, $\boldsymbol{z = c}$, which are therefore the *Eqs. of the point.*

It is most convenient to think XY *horizontal*, *right* or *east* being $+X$, *forward* or *north* $+Y$, as in Plane Geometry. Either up or down may be taken as + on Z, but *up* is better, according to the convention, important in Mechanics: *That side of a plane is + whence positive rotation* (as from $+X$ to $+Y$) *appears counter-clockwise.*

Clearly Space is cut by the three planes into eight *regions*. The *upper* four we name 1, 2, 3, 4, from the quadrants in XY on which they stand; those below, in the same order, 5, 6, 7, 8.

$-, +, \pm$	$+, +, \pm$
$-, -, \pm$	$+, -, \pm$

Then the signs of x, y, z in the eight regions are, as in the diagram, the *lower* sign of the z referring to the *lower* region.

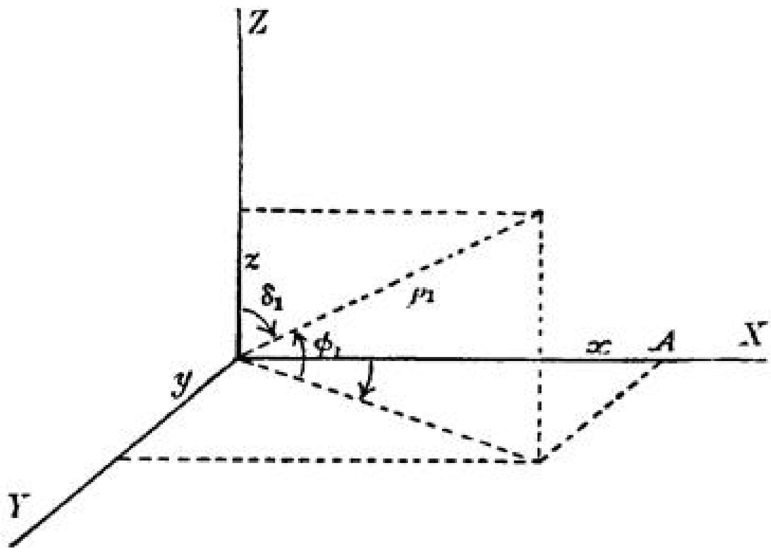
The ∡s $\widehat{yz}$, $\widehat{zx}$, $\widehat{xy}$ may be denoted by χ, ψ, ω; unless otherwise stated they will be considered right ∡s. We may call x, y, z *triplanar Cds.*, and speak of the *point* (x, y, z).

2. Around any RL. (say Z) as axis, suppose laid a cylinder of radius $r = r_1$. *All* points of the surface, and *no others*, are distant r_1 from the axis, and the surface is defined completely by its Eq. $r=r_1$. Through the axis pass a half-plane sloped $\theta=\theta_1$ to some *base*-plane through the axis (say ZX); then is this half-plane defined completely by its Eq., $\theta=\theta_1$. For *all* points on the RL., the intersection of half-plane and cylinder, and for *no others*, hold the relations: $r=r_1$, $\theta=\theta_1$, which are therefore the *Eqs. of that RL.* Pass a plane $\perp$ to Z, hence $\parallel$ to XY, and distant $z=z_1$ from this latter. By Art. 1, $z=z_1$ is its Eq. For the intersection of this plane and the RL. (r_1, θ_1), and for no other, hold the relations $r=r_1$, $\theta=\theta_1$, $z=z_1$, which are therefore the *Eqs. of the point*. We may call r, θ, z *cylindric Cds.*, and speak of the point (r, θ, z); r may always be taken $+$, $\theta+$ when reckoned counter-clockwise, $z+$ when reckoned up. Connecting the two systems of Cds., the relations hold:

$$x=r\cos\theta, \quad y=r\sin\theta, \quad z=z.$$

3. About any point (say the *origin* O) lay a sphere of radius ρ_1; clearly its Eq. is $\rho=\rho_1$. Pass a *half-plane* as in Art. 2; the Eqs. of the half great circle in which the half-plane meets the sphere are clearly $\rho=\rho_1$, $\theta=\theta_1$.

About Z lay a *cone* sloped δ_1 to Z and ϕ_1 to XY, so that $\delta_1+\phi_1=90°$; its Eq. is $\delta=\delta_1$, or $\phi=\phi_1$. For the *point* where it meets the half-circle, and for *no other*, hold the



relations: $\rho=\rho_1$, $\theta=\theta_1$, $\phi=\phi_1$ (or $\delta=\delta_1$), which are therefore *its Eqs.*

We may call ρ, θ, ϕ (or δ) *polar* or *spheric Cds.*, and speak of the *point* (ρ, θ, ϕ); ρ may be taken *always* $+$, $\theta+$ when reckoned *counter-clockwise*, $\phi+$ where reckoned *toward* $+Z$ (δ always $+$). Clearly θ and ϕ correspond to geographic *Longitude* and *Latitude*: δ may be called (north) *polar distance*.

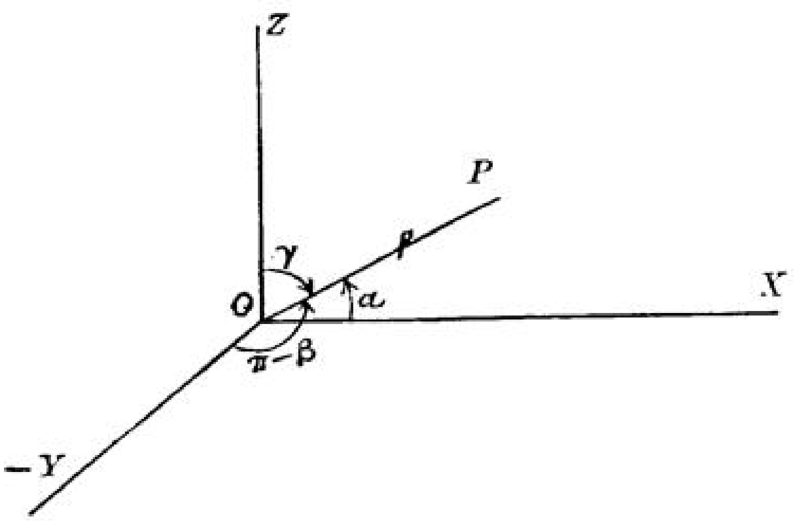
On projecting ρ on Z and on XY, and this last projection on X and Y, these relations become manifest:

$$x = \rho \cos\phi \cos\theta = \rho \sin\delta \cos\theta,$$
$$y = \rho \cos\phi \sin\theta = \rho \sin\delta \sin\delta,$$
$$z = \rho \sin\phi \qquad\quad = \rho \cos\delta.$$

4. Hereafter, *cosine* resp. *sine* may be denoted by putting a horizontal bar *under* resp. vertical bar *after* the argument, thus:

$$\omega| = \sin\omega, \qquad \underline{\omega} = \cos\omega.$$

Call the tract OP from the origin O to any point P the *radius vector* of the point, and denote it by ρ; denote its slopes to



X, Y, Z by α, β, γ, and call their cosines $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$ **direction-cosines** of ρ. Then, by definition, the projections of ρ on the axes are the Cds. of P; i.e.,

$$x = \underline{\alpha}\rho, \qquad y = \underline{\beta}\rho, \qquad z = \underline{\gamma}\rho.$$

Squaring, adding, and remembering $x^2 + y^2 + z^2 = \rho^2$, we get

$$\underline{\alpha}^2 + \underline{\beta}^2 + \underline{\gamma}^2 = 1. \qquad \text{(A)}$$

For one factor in each term put $\frac{x}{\rho}$, $\frac{y}{\rho}$, $\frac{z}{\rho}$, multiply by ρ, and get

$$\underline{\alpha}x + \underline{\beta}y + \underline{\gamma}z = \rho, \qquad \text{(B)}$$

which simply says the *sum of the projections of a train of tracts from* **O** *to* **P** *on OP is OP*, as is already known.

Now take a plane perpendicular to OP; it will be sloped α, β, γ to YZ, ZX, XY; let it meet X, Y, Z at A, B, C, and call the area of the $\triangle\ ABC\ \Delta$; the projections of this $\triangle$ on YZ, ZX, XY are the $\triangle$s BOC, COA, AOB; their areas are $\underline{\alpha}\Delta$, $\underline{\beta}\Delta$, $\underline{\gamma}\Delta$; squaring, we get $\overline{\underline{\alpha}\Delta}^2 + \overline{\underline{\beta}\Delta}^2 + \overline{\underline{\gamma}\Delta}^2 = \Delta^2$ since $\underline{\alpha}^2 + \underline{\beta}^2 + \underline{\gamma}^2 = 1$; i.e., *the squared hypotenuse-face of a right-sided tetraeder equals the sum of the squares of the other faces.* This proposition is the analogue (for space) of the Pythagorean.

The distance d between P_1 $(\rho_1, \alpha_1, \beta_1, \gamma_1)$ and P_2 $(\rho_2, \alpha_2, \beta_2, \gamma_2)$ is plainly the diagonal of a parallelepiped, whose edges are $x_1 - x_2$, $y_1 - y_2$, $z_1 - z_2$; hence $d^2 = \overline{x_1 - x_2}^2 + \overline{y_1 - y_2}^2 + \overline{z_1 - z_2}^2$.

By the *Law of Cosines* $\quad d^2 = \rho_1{}^2 + \rho_2{}^2 - 2\rho_1\rho_2\, \underline{\widehat{\rho_1\rho}}$.
Hence,

$$\underline{\widehat{\rho_1\rho_2}} = (x_1x_2 + y_1y_2 + z_1z_2) : \rho_1\rho_2 = \underline{\alpha}_1\underline{\alpha}_2 + \underline{\beta}_1\underline{\beta}_2 + \underline{\gamma}_1\underline{\gamma}_2 \qquad (C)$$

This last expression for the *cosine of the* ∡ between two RLs. in terms of their *direction-cosines* holds even when the RLs. do *not* meet, since *the* ∡ *between two non-intersecting RLs. equals the* ∡ *between two* ∥ *intersecting RLs.*

Corollary. *When, and only when,* ρ_1 and ρ_2 are $\perp$,

$$\underline{\alpha}_1\underline{\alpha}_2 + \underline{\beta}_1\underline{\beta}_2 + \underline{\gamma}_1\underline{\gamma}_2 = 0. \qquad (C_1)$$

The ∡ between *two planes* equals the (adjacent) ∡ between *two* ⊥*s* on the planes, but the *slope* of a RL. to a plane is the *complement of the slope* of the RL. to a ⊥ on the plane; hence, the *slopes to the axes* of a plane ⊥ to ρ are $\mathrm{A} = 90^\circ - \alpha$, $\mathrm{B} = 90^\circ - \beta$, $\Gamma = 90^\circ - \gamma$; hence,

$$\mathrm{A}|^2 + \mathrm{B}|^2 + \Gamma|^2 = 1 \text{ and } \widehat{\Pi_1\Pi_2} = \mathrm{A}_1|\mathrm{A}_2| + \mathrm{B}_1|\mathrm{B}_2| + \Gamma_1|\Gamma_2| \qquad (C_2)$$

4.* In case of oblique axes we use the theorem: the projection on any RL. of a tract between two points equals the sum of the projections of any train of tracts between the points; hence

$$\rho\,.\,\underline{\widehat{\rho l}} = x\,.\,\underline{\widehat{xl}} + y\,.\,\underline{\widehat{yl}} + z\,.\,\underline{\widehat{zl}}.$$

Take as l, in turn, the vector ρ and X, Y, Z; so we get

$$\left.\begin{array}{ll} \rho = x\,.\,\underline{\widehat{x\rho}} + y\,.\,\underline{\widehat{y\rho}} + z\,.\,\underline{\widehat{z\rho}}, & \rho\,.\,\underline{\widehat{x\rho}} = x + y\,.\,\underline{\widehat{yx}} + z\,.\,\underline{\widehat{zx}} \\ \rho\,.\,\underline{\widehat{y\rho}} = x\,.\,\underline{\widehat{xy}} + y + z\,.\,\underline{\widehat{zy}}, & \rho\,.\,\underline{\widehat{z\rho}} = x\,.\,\underline{\widehat{xz}} + y\,.\,\underline{\widehat{yz}} + z. \end{array}\right\} \qquad \text{(D)}$$

On multiplying in turn by ρ, x, y, z, and adding, there results

$$\rho^2 = x^2 + y^2 + z^2 + 2\,yz\,.\,\underline{\widehat{yz}} + 2\,zx\,.\,\underline{\widehat{zx}} + 2\,xy\,.\,\underline{\widehat{xy}}.$$

Hence $\underline{\widehat{x\rho}}, \underline{\widehat{y\rho}}, \underline{\widehat{z\rho}}$, or $\underline{\alpha}, \underline{\beta}, \underline{\gamma}$, are found at once by using (D).

To express conversely x, y, z through $\rho, \alpha, \beta, \gamma$, form the determinant $\triangle$ of the four Eqs., remembering $\widehat{yz} = \chi, \quad \widehat{zx} = \psi, \quad \widehat{xy} = \omega$;

$$\Delta = -\begin{vmatrix} 1 & \underline{\alpha} & \underline{\beta} & \underline{\gamma} \\ \underline{\alpha} & 1 & \underline{\omega} & \underline{\psi} \\ \underline{\beta} & \underline{\omega} & 1 & \underline{\chi} \\ \underline{\gamma} & \underline{\psi} & \underline{\chi} & 1 \end{vmatrix}, \quad \text{and put} \quad S^2 = \begin{vmatrix} 1 & \underline{\omega} & \underline{\psi} \\ \underline{\omega} & 1 & \underline{\chi} \\ \underline{\psi} & \underline{\chi} & 1 \end{vmatrix};$$

then, denoting the co-factors in $\triangle$ by like letters accented,

$$x = \alpha'\rho : S^2, \qquad y = \beta'\rho : S^2, \qquad z = \gamma'\rho : S^2.$$

On putting these values in the first Eq. of (D) and clearing, there results

$$\chi|^2\underline{\alpha}^2 + \psi|^2\underline{\beta}^2 + \omega|^2\,\underline{\gamma}^2 - 2\,\chi''\underline{\beta}\underline{\gamma} - 2\,\psi''\underline{\gamma}\underline{\alpha} - 2\,\omega''\underline{\alpha}\underline{\beta} = S^2, \qquad \text{(A*)}$$

where χ'', ψ'', ω'' are co-factors of like letters in S^2.

To find the distance d between two points, P and P_1, take P_1 as a new origin (see Art. 6), then the Cds. of P are $x - x_1, y - y_1, z - z_1$; put these for x, y, z, and d for ρ in the formula found.

To find the cosine of the $\measuredangle\, POP_1$, put ρ_1 for l; then

$$\rho\,.\,\underline{\widehat{\rho\rho_1}} = x\,.\,\underline{\widehat{x\rho_1}} + y\,.\,\underline{\widehat{y\rho_1}} + z\,.\,\underline{\widehat{z\rho_1}};$$

on substituting for the cosines on the right, there results

$$\underline{\widehat{\rho\rho_1}} = \{xx_1 + yy_1 + zz_1 + \underline{\chi}(yz_1 + zy_1) + \underline{\psi}(zx_1 + xz_1) + \underline{\omega}(xy_1 + yx_1)\} : \rho\rho_1. \quad (\mathrm{C_1}^*)$$

On comparing (A) with (A*) and ($\mathrm{C_1}$) with ($\mathrm{C_1}$*), analogy would suggest the following Eq. as ($\mathrm{C_2}$*):

$$\overset{\frown}{\underline{\rho\rho_1}} = \chi|^2\underline{\alpha\alpha_1} + \psi|^2\underline{\beta\beta_1} + \omega|^2\underline{\gamma\gamma_1} - \chi''(\underline{\beta\gamma_1} + \underline{\beta_1\gamma})$$
$$- \psi''(\underline{\gamma\alpha_1} + \underline{\alpha\gamma_1}) - \omega''(\underline{\alpha\beta_1} + \underline{\beta\alpha_1}) : S^2. \qquad (C_2{}^*)$$

This conjecture is readily confirmed thus: on putting, in ($C_1{}^*$), for the Cds. their values found above, the product $\rho\rho_1$ vanishes, S^4 becomes divisor, subscribed and unsubscribed letters (α, β, γ) combine every way in sets of two, and the result, symmetric as to the subscribed and unsubscribed letters, since $\overset{\frown}{\underline{\rho\rho_1}} = \overset{\frown}{\underline{\rho_1\rho}}$, is of the form

$$\{A\underline{\alpha\alpha_1} + B\underline{\beta\beta_1} + C\underline{\gamma\gamma_1} + A'(\underline{\beta\gamma_1} + \underline{\gamma\beta_1}) + B'(\underline{\gamma\alpha_1} + \underline{\alpha\gamma_1})$$
$$+ C'(\underline{\alpha\beta_1} + \underline{\beta\alpha_1})\} = S^4 . \overset{\frown}{\underline{\rho\rho_1}},$$

where A, B, C, A', B', C' depend *only* on χ, ψ, ω.

For $\overset{\frown}{\underline{\rho\rho_1}} = 0$, this must pass over into (A*), for then $\alpha = \alpha_1$, etc.; hence $A = \chi|^2 . S^2$, etc., which on substitution yield $C_2{}^*$, as guessed.

We have seen that in case of rectang. axes

$$\overline{BOC}^2 + \overline{COA}^2 + \overline{AOB}^2 = \overline{ABC}^2,$$

i.e., *the squared area of a* $\triangle$ (and hence of any plane figure) *equals the sum of the squared areas of its projections on three rectangular planes.* To find the general theorem for oblique axes, of which the above is a special case, we note that the corresponding formulae for rectang. and oblique axes are, when $\measuredangle AOB = \omega$,

$$\overline{AB}^2 = \overline{OA}^2 + \overline{OB}^2 \quad \text{and} \quad \overline{AB}^2 = \overline{OA}^2 + \overline{OB}^2 - 2\,OA\,.\,OB\,.\,\underline{\omega}.$$

On putting $OA = a$, $OB = b$, these formulae may be written

$$\overline{AB}^2 = -\begin{vmatrix} 0 & a & b \\ a & 1 & 0 \\ b & 0 & 1 \end{vmatrix}, \qquad \overline{AB}^2 = -\begin{vmatrix} 0 & a & b \\ a & 1 & \underline{\omega} \\ b & \underline{\omega} & 1 \end{vmatrix}.$$

So, too, putting $OC = c$, we have in case of rectang. axes

$$4\,\overline{ABC}^2 = -\begin{vmatrix} 0 & bc & ca & ab \\ bc & 1 & 0 & 0 \\ ca & 0 & 1 & 0 \\ ab & 0 & 0 & 1 \end{vmatrix}; \quad \text{whence} \quad 4\,\overline{ABC}^2 = -\begin{vmatrix} 0 & bc & ca & ab \\ bc & 1 & \underline{\omega} & \underline{\psi} \\ ca & \underline{\omega} & 1 & \underline{\chi} \\ ab & \underline{\psi} & \underline{\chi} & 1 \end{vmatrix},$$

if the analogy holds, a result easily verified.

The *area of a parallelogram* whose sides, a and b, are sloped ω is $ab \sin \omega$, or

$$ab\begin{vmatrix} 1 & \underline{\omega} \\ \underline{\omega} & 1 \end{vmatrix}^{\frac{1}{2}}.$$

Accordingly we might suspect the volume of a parallelepiped whose edges a, b, c are sloped ω, χ, ψ to each other to be

$$abc \begin{vmatrix} 1 & \underline{\omega} & \underline{\psi} \\ \underline{\omega} & 1 & \underline{\chi} \\ \underline{\psi} & \underline{\chi} & 1 \end{vmatrix}^{\frac{1}{2}}.$$

This conclusion from analogy is readily verified thus: take the parallelogram $ab \,.\, \omega|$ as base, project C on X at A' and on XY at C' (the edges being taken as axes); calling the diedral χ along XA, we have

$$CA' = c\psi|, \qquad CC' = c \,.\, \psi| \,.\, A|.$$

By spheric trigonometry $\quad A| = \sqrt{1 - \underline{\chi}^2 - \underline{\psi}^2 - \underline{\omega}^2 + 2\,\underline{\chi}\underline{\psi}\underline{\omega}} : \psi| \,.\, \omega|$; the volume is $ab \,.\, \omega| \,.\, CC'$, whence the formula above.

The radical of the determinant, which is $\boldsymbol{S}$ of A*, is thus seen to be the *volume of a parallelepiped of unit edges*, sloped ω, χ, ψ to each other. Since the factor $\boldsymbol{S}$ turns the *product* of the edges into the *volume* of the parallelepiped, just as $\sin \widehat{ab}$ turns the *product* of the sides into the *area* of the parallelogram, it has been named (by Staudt) *sine* of the *solid angle* of the edges, and may be written $\sin \widehat{abc}$ or $\widehat{abc}|$.

If ξ, η, ζ be the *slopes* of X to YZ, Y to ZX, Z to XY, then

$$CC' = c \,.\, \xi|;$$

but $\quad CC' = c \,.\, \psi| \,.\, A| = c \,.\, \widehat{xyz}| : \omega|; \quad$ hence $\quad \widehat{xyz}| = \zeta| \,.\, \omega|, \quad$ or

$$\boldsymbol{S} = \xi| \,.\, \chi| = \eta| \,.\, \psi| = \zeta| \,.\, \omega|.$$

To find the area of any $\triangle$ (or other plane figure) *in terms of its projections*, call these projections $I \,.\, \widehat{yz}|,\ J \,.\, \widehat{zx}|,\ K \,.\, \widehat{xy}|$ (or $I \,.\, \chi|,\ J \,.\, \psi|,\ K \,.\, \omega|$), and suppose the $\perp p$ on the plane of the $\triangle$ directed by α, β, γ; then

$$\triangle \,.\, \underline{\alpha} = I\chi| \,.\, \xi|, \quad \text{or} \quad \triangle \,.\, \underline{\alpha} = I \,.\, \boldsymbol{S},$$

since each is the projection of $\triangle$ on a plane $\perp$ to X. Hence

$$\triangle \,.\, \underline{\alpha} = I \,.\, \boldsymbol{S}, \quad \triangle \,.\, \underline{\beta} = J \,.\, \boldsymbol{S}, \quad \triangle \,.\, \underline{\gamma} = K \,.\, \boldsymbol{S}.$$

If h be the *altitude* and l the *edge* of a prism standing on $\triangle$ as base, and i, j, k be the projections of l on X, Y, Z, then $\quad h = i\underline{\alpha} + j\underline{\beta} + k\underline{\gamma}; \quad$ hence

$$\triangle \,.\, h = \text{volume of prism} = (iI + jJ + kK)\boldsymbol{S}.$$

The *projections* of a *tract* on the *axes*, ‖ to the Cd. planes, resp. the *projections* of a *plane* figure on the Cd. *planes*, ‖ to the axes, are called *Cds.* of the *tract* resp. *plane*.

Now take the pyramid whose vertex is 0, whose base is PP_1P_2; it is $\frac{1}{3}$ of the prism with base OP_1P_2 and edge OP; the Cds. of the edge and the double Cds. of the base of this prism are x, y, z, and

$$\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} |\chi|, \quad \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} |\psi|, \quad \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} |\omega|.$$

Hence by the above formula we have

$$6\,OPP_1P_2 = \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \mathbf{S}.$$

To move the origin to (x_3, y_3, z_3), it suffices to put $x - x_3$ for x, $y - y_3$ for y, $z - z_3$ for z, etc., and write

$$6\,OPP_1P_2 = 6\,PP_1P_2P_3 = \begin{vmatrix} 0 & x - x_3 & y - y_3 & z - z_3 \\ 0 & x_1 - x_3 & y_1 - y_3 & z_1 - z_3 \\ 0 & x_2 - x_3 & y_2 - y_3 & z_2 - z_3 \\ 1 & x_3 & y_3 & z_3 \end{vmatrix} \mathbf{S}.$$

On adding the last row to each of the others there results

$$6\,PP_1P_2P_3 = \begin{vmatrix} 1 & x & y & z \\ 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \end{vmatrix} \mathbf{S} = 6\,T.$$

When, and only when, this six-fold tetraedral volume is 0, does the point $P(x, y, z)$ lie in the plane of $P_1P_2P_3$; hence $\mathbf{T} = 0$ is the *Eq. of the plane through the three points P_1, P_2, P_3.*

5. By projecting a point ‖ to Z (say), its X and Y are *not changed;* i.e., *the x and y of a* point *are the x and y of its* XY-projection, and are the same for all points of a RL. ‖ to Z. To find, then, the x and y of a point cutting a tract P_1P_2 in ratio $n_1 : n_2$, project the tract on XY; the Cds. of the projection are the Cds. sought:

$$\boldsymbol{x} = \frac{n_1 x_2 + n_2 x_1}{n_1 + n_2}, \quad \boldsymbol{y} = \frac{n_1 y_2 + n_2 y_1}{n_1 + n_2}, \quad \text{and so} \quad \boldsymbol{z} = \frac{n_1 z_2 + n_2 z_1}{n_1 + n_2}.$$

Transformation of Co-ordinates.

6. For *pushing* the axes, *not changing* their directions, clearly

$$x = x' + x_0, \quad y = y' + y_0, \quad z = z' + z_0.$$

If the axes (rectangular) be *turned* from the position X, Y, Z into the position X', Y', Z', so that the $\measuredangle$s $\widehat{XX'}$, $\widehat{XY'}$, $\widehat{XZ'}$, are α', α'', α''', then the sum of the projections of x', y', z' on X, for any point, is simply the x of that point; or

and so

$$\left.\begin{aligned} x &= x'\underline{\alpha}' + y'\underline{\alpha}'' + z'\underline{\alpha}'''; \\ y &= x'\underline{\beta}' + y'\underline{\beta}'' + z'\underline{\beta}''', \\ z &= x'\underline{\gamma}' + y'\underline{\gamma}'' + z'\underline{\gamma}'''. \end{aligned}\right\} \qquad \text{(E)}$$

The nine $\measuredangle$s are clearly *not all* at will, since, if α' and β' be chosen, γ' is thereby fixed; for all RLs. sloped α' to X lie on a cone about X, and all sloped β' to Y on a cone about Y, and X' must be a common RL. of these two cones; the axis X' being fixed, so is the plane $Y'Z'$, and the choice of one more $\measuredangle$ fixes Y' and Z'. So *only three* of the nine can be chosen at will; hence there must hold *six Eqs. of condition* among the nine $\measuredangle$s. These are

$$\begin{aligned} &\underline{\alpha}'^2 + \underline{\beta}'^2 + \underline{\gamma}'^2 = 1, && \underline{\alpha}' \cdot \underline{\alpha}'' + \underline{\beta}' \cdot \underline{\beta}'' + \underline{\gamma}' \cdot \underline{\gamma}'' = 0, \\ &\underline{\alpha}''^2 + \underline{\beta}''^2 + \underline{\gamma}''^2 = 1, && \underline{\alpha}'' \cdot \underline{\alpha}''' + \underline{\beta}'' \cdot \underline{\beta}''' + \underline{\gamma}'' \cdot \underline{\gamma}''' = 0, \\ &\underline{\alpha}'''^2 + \underline{\beta}'''^2 + \underline{\gamma}'''^2 = 1; && \underline{\alpha}''' \cdot \underline{\alpha}' + \underline{\beta}''' \cdot \underline{\beta}' + \underline{\gamma}''' \cdot \underline{\gamma}' = 0. \end{aligned}$$

The first three say X, Y, Z are rectangular, the second three say X', Y', Z' are rectangular, as appears from (A) and (B).

The formulæ for passing from X', Y', Z' to X, Y, Z are plainly

$$\begin{aligned} x' &= x\underline{\alpha}' + y\underline{\beta}' + z\underline{\gamma}', \\ y' &= x\underline{\alpha}'' + y\underline{\beta}'' + z\underline{\gamma}'', \\ z' &= x\underline{\alpha}''' + y\underline{\beta}''' + z\underline{\gamma}'''. \end{aligned}$$

Accordingly there must hold these *six Eqs. of condition:*

$$\begin{aligned} &\underline{\alpha}'^2 + \underline{\alpha}''^2 + \underline{\alpha}'''^2 = 1, && \underline{\alpha}' \cdot \underline{\beta}' + \underline{\alpha}'' \cdot \underline{\beta}'' + \underline{\alpha}''' \cdot \underline{\beta}''' = 0, \\ &\underline{\beta}'^2 + \underline{\beta}''^2 + \underline{\beta}'''^2 = 1, && \underline{\beta}' \cdot \underline{\gamma}' + \underline{\beta}'' \cdot \underline{\gamma}'' + \underline{\beta}''' \cdot \underline{\gamma}''' = 0, \\ &\underline{\gamma}'^2 + \underline{\gamma}''^2 + \underline{\gamma}'''^2 = 1; && \underline{\gamma}' \cdot \underline{\alpha}' + \underline{\gamma}'' \cdot \underline{\alpha}'' + \underline{\gamma}''' \cdot \underline{\alpha}''' = 0. \end{aligned}$$

These six Eqs. must then be *equivalent* to the first six; this is clear geometrically, and may be proved analytically thus:

Form the determinant

$$C = \begin{vmatrix} \underset{-}{\alpha}' & \underset{-}{\alpha}'' & \alpha''' \\ \underset{-}{\beta}' & \underset{-}{\beta}'' & \beta''' \\ \underset{-}{\gamma}' & \underset{-}{\gamma}'' & \gamma''' \end{vmatrix};$$

by solving the first three Eqs. directly, we get

$$C\cdot x' = \mathrm{A}'x + \mathrm{B}'y + \Gamma'z, \qquad \text{and so on};$$

hence, $\mathrm{A}' = \underset{-}{\alpha}'C$, $\mathrm{B}' = \underset{-}{\beta}'C$, and so on; now

$$\underset{-}{\alpha}'\cdot\mathrm{A}' + \underset{-}{\alpha}''\cdot\mathrm{A}'' + \underset{-}{\alpha}'''\cdot\mathrm{A}''' = C, \quad \text{and five other like Eqs.,}$$

while $\underset{-}{\alpha}'\cdot\mathrm{B}' + \underset{-}{\alpha}''\cdot\mathrm{B}'' + \underset{-}{\alpha}'''\cdot\mathrm{B}''' = 0$, and five other like Eqs.;

whence, on replacing the co-factors, A', etc., result the twelve Eqs.

By squaring C according to the *Multiplication Theorem* of Determinants, it is shown that $C^2 = 1$; hence $\mathbf{C = \pm 1}$, according, namely, as $X'Y'Z'$ is *congruent* or *symmetric* with XYZ; i.e., according as, when $+X'$ falls on $+X$ and $+Y'$ on $+Y$, $+Z'$ falls on $+Z$ or $-Z$.

The formulæ (E) hold even when X', Y', Z' are not rectangular, since this rectangularity was not assumed in their deduction.

The general formulæ for oblique axes are found, precisely as in Plane Geometry, to be

$$x\,.\,(\widehat{x, yz})| = x'\,.\,(\widehat{x', yz})| + y'\,.\,(\widehat{y', yz})| + z'\,.\,(\widehat{z', yz})|,$$

and two got by permuting x, y, z. The nine coefficients are again connected by six Eqs. of condition.

Note that the Eqs. of Transformation are *linear* in Cds.

EXERCISE.

Show that $\widehat{\rho_1\rho_2}|^2 = |\alpha_1\beta_2|^2 + |\beta_1\gamma_2|^2 + |\gamma_1\alpha_2|^2$.

The Plane and the Right Line.

7. *A single Eq. in x, y, z represents a surface.* For we may assign all real pairs of values to x and y (say), and reckon the corresponding values of z. The pairs (x, y) fix points in XY; over each such point suppose the fixed value of z erected, laid off parallel to Z; the ends of all such z's will lie on and determine the surface, which of course may be real or imaginary, continuous or discontinuous.

Two Eqs. in x, y, z represent a line. For all points whose Cds. satisfy both Eqs. must lie on both surfaces picturing the two Eqs., hence must lie on the intersection of the surfaces, i.e., on a line.

Three Eqs. in x, y, z of mth, nth, resp. pth degree, represent mnp fixed points. For the three Eqs. are fulfilled at once by *mnp triplets* of values, each pictured by a point.

Transformation of Cds. does not change the degree of the Eq. in x, y, z. For the Eqs. of transformation are linear.

8. A *line* is fixed by the Eqs. of any *two surfaces* through it; the simplest surfaces are generally *two cylinders* || *each to an axis.* The Eqs. of these cylinders are the same as the Eqs. of their intersections, each with the plane of the other two axes, since the third Cd. is the same for every point of any given element of one of them. They are clearly the *projecting cylinders* of the line, and their intersections with the planes are the *projections* of the line on those planes. Hence, as *Eqs. of a* **line** may be taken the *Eqs. of any two of its projections*, on the planes of two axes, || to the third axis.

If the line be a RL., the projecting cylinders are *planes*, and the *projections* are RLs. whose Eqs. may be written

$$y = sx + b', \qquad y = tz + b, \qquad x = uz + a,$$

any two of which (generally the last) may be taken as **Eqs. of the RL.** Clearly, a, b', b are the intercepts of projections on X, Y, while s, t, u are direction-coefficients.

Symmetric Eqs. of the RL. may be got thus: Be (x_1, y_1, z_1) and (x, y, z) a fixed and a variable point on the RL., $\boldsymbol{d}$ their distance apart, and let its direction-cosines be $\underline{\alpha}$, $\underline{\beta}$, γ; then

$$d = \frac{x - x_1}{\underline{\alpha}} = \frac{y - y_1}{\underline{\beta}} = \frac{z - z_1}{\underline{\gamma}},$$

or $$x = x_1 + \underline{\alpha} d, \quad y = y_1 + \underline{\beta} d, \quad z = z_1 + \underline{\gamma} d.$$

In $$\frac{x - x_1}{\underline{\alpha}} = \frac{y - y_1}{\underline{\beta}} = \frac{z - z_1}{\underline{\gamma}}$$

instead of $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$, may be put any three *proportionates*, as λ, μ, ν, so that

$$\frac{x - x_1}{\lambda} = \frac{y - y_1}{\mu} = \frac{z - z_1}{\nu}.$$

If $\lambda : \underline{\alpha} = \mu : \underline{\beta} = \nu : \underline{\gamma} = f$, then $f = \sqrt{\lambda^2 + \mu^2 + \nu^2}$.

On comparing the *symmetric* and the *tangential* forms, we see

$$s = \underline{\beta} : \underline{\alpha}, \quad t = \underline{\beta} : \underline{\gamma}, \quad u = \underline{\alpha} : \underline{\gamma};$$

whence $u = t : s$.

If the RL. goes through (x_2, y_2, z_2), then

$$\frac{x_2 - x_1}{\underline{\alpha}} = \frac{y_2 - y_1}{\underline{\beta}} = \frac{z_2 - z_1}{\underline{\gamma}};$$

i.e., $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$, are proportional to $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$; hence

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1},$$

Eq. of a RL. through two points.

9. Two RLs. in space may or may not meet; *if they do*, the z of the intersection of their projections on YZ must be the same as the z of the intersection of their projections on ZX: i.e., if $y = t_1 z + b_1$, $x = u_1 z + a_1$, and $y = t_2 z + b_2$, $x = u_2 z + a$ be the RLs., then

$$\overline{b_1 - b_2} : \overline{t_1 - t_2} = \overline{a_1 - a_2} : \overline{u_1 - u_2}.$$

Or the four Eqs. must hold at once; this yields the same result in a determinant.

If two RLs., l_1 and l_2, are directed by $\underline{\alpha}_1, \underline{\beta}_1, \underline{\gamma}_1$ and $\underline{\alpha}_2, \underline{\beta}_2, \underline{\gamma}_2$, or by their proportionals λ_1, μ_1, ν_1 and λ_2, μ_2, ν_2, then

$$\underline{\widehat{l_1 l_2}} = \underline{\alpha}_1\underline{\alpha}_2 + \underline{\beta}_1\underline{\beta}_2 + \underline{\gamma}_1\underline{\gamma}_2 = \frac{\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2}{\sqrt{\lambda_1^2 + \mu_1^2 + \nu_1^2}\cdot\sqrt{\lambda_2^2 + \mu_2^2 + \nu_2^2}}$$
$$= \frac{1 + t_1t_2 + u_1u_2}{\sqrt{1 + t_1^2 + u_1^2}\cdot\sqrt{1 + t_2^2 + u_2^2}}$$

If the RLs. be perpendicular, then

$$\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = 0 = 1 + t_1t_2 + u_1u_2.$$

If the RL. (t, u) be $\perp$ to (t', u'), then

$$1 + tt' + uu' = 0\,; \qquad (1)$$

if it goes through (x_1, y_1, z_1), then

$$y_1 = tz_1 + b, \quad x_1 = uz_1 + a\,; \qquad (2)$$

if it meets the RL. (t', u'), then

$$\overline{b - b'} : \overline{t - t'} = \overline{a - a'} : \overline{u - u'}. \qquad (3)$$

These four Eqs. fix the values of t, u, a, b. They may be found thus:

From (2), $a = x_1 - uz_1$, $b = y_1 - tz_1$; put these into (3), whence

$$(u'z_1 + a' - x_1)t - (t'z_1 + b' - y_1)u = (a' - x_1)t' - (b' - y_1)u'. \qquad (4)$$

From (1) and (4) can now be found t and u, thus:

Put
$$N = u'(u'z_1 + a' - x_1) + t'(t'z_1 + b' - y_1),$$
$$M = t'\{(b' - y_1)u' - (a' - x_1)t'\} - (u'z_1 + a' - x_1),$$
$$L = u'\{(a' - x_1)t' - (b' - y_1)u'\} - (t'z_1 + b' - y_1)\,;$$

then
$$t = M : N, \qquad u = L : N\,;$$
$$a = x_1 - \frac{L}{N}z_1, \quad b = y_1 - \frac{M}{N}z_1.$$

Accordingly,

$$\overline{x - x_1} : L = \overline{y - y_1} : M = \overline{z - z_1} : N$$

is the Eq. of the $\perp$ from (x_1, y_1, z_1) on the RL. $y = tz + b$, $x = uz + a$.

The student may now easily show that the *Cds. of the intersection* are

$$x_1 - (L : \overline{1 + t'^2 + u'^2}),$$
$$y_1 - (M : \overline{1 + t'^2 + u'^2}),$$
$$z_1 - (N : \overline{1 + t'^2 + u'^2}),$$

while the *distance from* (x_1, y_1, z_1) *to the intersection is*

$$\sqrt{L^2 + M^2 + N^2} : \overline{1 + t'^2 + u'^2};$$

or, after simplification,

$$\sqrt{\{(\overline{a' - x_1}\, t' - \overline{b' - y_1}\, u')^2 + (u'z_1 + a' - x_1)^2 + (t'z_1 + b' - y_1)^2\}}$$
$$: \sqrt{1 + t'^2 + u'^2}.$$

10. *If the RL.* $y = tz + b, \quad x = uz + a,$ *be* $\perp$ *to the two RLs.*

$$y = t_1 z + b_1, \quad x = u_1 z + a_1,$$

and $$y = t_2 z + b_2, \quad x = u_2 z + a_2,$$

then $$t_1 t + u_1 u + 1 = 0, \quad t_2 t + u_2 u + 1 = 0,$$

whence $$t = \overline{u_1 - u_2} : \overline{t_1 u_2 - t_2 u_1}, \quad u = \overline{t_1 - t_2} : \overline{u_1 t_2 - u_2 t_1}.$$

If the RLs. be directed by (α, β, γ), $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, then

$$\underline{\alpha}_1 \underline{\alpha} + \underline{\beta}_1 \underline{\beta} + \underline{\gamma}_1 \underline{\gamma} = 0,$$
$$\underline{\alpha}_2 \underline{\alpha} + \underline{\beta}_2 \underline{\beta} + \underline{\gamma}_2 \underline{\gamma} = 0,$$
$$\underline{\alpha}^2 + \underline{\beta}^2 + \underline{\gamma}^2 = 1.$$

From the first two Eqs., on solving as to $\underline{\alpha} : \underline{\gamma}$ and $\underline{\beta} : \underline{\gamma}$, it results that

$$\underline{\alpha} : \underline{\beta} : \underline{\gamma} = \underline{\beta}_1 \underline{\gamma}_2 - \underline{\beta}_2 \underline{\gamma}_1 : \underline{\gamma}_1 \underline{\alpha}_2 - \underline{\alpha}_1 \underline{\gamma}_2 : \underline{\alpha}_1 \underline{\beta}_2 - \underline{\beta}_1 \underline{\alpha}_2.$$

If the RL. (t, u) also meets the RLs. (t_1, u_1), (t_2, u_2), then

$$(b - b_1)(u - u_1) = (a - a_1)(t - t_1)$$

and $$(b - b_2)(u - u_2) = (a - a_2)(t - t_2).$$

Form the determinant

$$D = \begin{vmatrix} 1 & 1 & 1 \\ u & u_1 & u_2 \\ t & t_1 & t_2 \end{vmatrix},$$

and denote its co-factors, as usual, by like large letters; then

$$bD = b_1U_1T_2 - b_2U_2T_1 + (a_1 - a_2)\,U_1U_2,$$

and so for aD.

Thus is determined *the common* ⊥ *to two RLs.*

The length of the intercept on it between the two may now be found by finding the intersections; but that is tedious.

11. The symmetric Eqs. of a RL. $\frac{x-x_1}{\lambda} = \frac{y-y_1}{\mu} = \frac{z-z_1}{\gamma}$, contain *six parameters or arbitraries*, $\lambda, \mu, \nu, x_1, y_1, z_1$, which might be called the **Cds.** *of the RL. in Space.* But they are *not independent*, since four arbitraries (*Cds.*), (t, u, a, b), fix the RL. First, λ, μ, ν are proportional to α, β, γ, and these are connected by the relation $\alpha^2 + \beta^2 + \gamma^2 = 1$; secondly, denoting $\mu z_1 - \nu y_1$, $\nu x_1 - \lambda z_1$, $\lambda y_1 - \mu x_1$ by Λ, M, N, we see the relation holds:

$$\lambda\Lambda + \mu\mathrm{M} + \nu\mathrm{N} = 0.$$

These six symbols, $\lambda, \mu, \nu, \Lambda$, M, N, thus connected by *two Eqs. of condition*, we may call **Cds.** of the RL.

The last three are interpreted geometrically later. (Art. 25.)

12. *The Eq. of a plane is of first degree in x, y, z.* For the Eq. of a plane ∥ to a *Cd.* plane, as XY, is $z = z_1$; this plane may be referred to any other system of axes by a linear transformation of *Cds.*, and such a transformation cannot change the degree.

Accordingly the Eq. of a plane is of the form

$$lx + my + nz + d = 0. \qquad (1)$$

If a, b, c be the intercepts on the axes, then clearly they equal $-d:l,\quad -d:m,\quad -d:n$; hence they vary *inversely* as l, m, n; the Eq. of the plane may also be written

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1. \tag{2}$$

Conversely, *an Eq. of first degree in x, y, z, represents a plane.* For, if d be not 0, it may be brought into form (2), which is known to represent a plane making intercepts a, b, c on X, Y, Z; if d be 0, push the origin out any distance, say d', on any axis, say X, by putting $x+d'$ for x in the Eq.; then the Eq. may be brought into the form (2), and the previous reasoning applies.

13. Drop a $\perp$ p from O on the plane, directed by α, β, γ; then $p=a\underline{\alpha}=b\underline{\beta}=c\underline{\gamma}$, and on substitution results

$$x\underline{\alpha}+y\underline{\beta}+z\underline{\gamma}-p=0, \tag{3}$$

the *Normal Eq.* of the plane, which we may write

$$\mathbf{N}=\mathbf{0}.$$

If F be the factor that turns the *general* into the *normal* form, then

$$Fl=\underline{\alpha},\quad Fm=\underline{\beta},\quad Fn=\underline{\gamma};$$

$$\therefore F^2(l^2+m^2+n^2)=1,$$

whence $$F=1:\sqrt{l^2+m^2+n^2},$$

and $$\underline{\boldsymbol{\alpha}}=\boldsymbol{l}:\sqrt{\boldsymbol{l}^2+\boldsymbol{m}^2+\boldsymbol{n}^2};$$

and so for $\underline{\beta}$ and $\underline{\gamma}$.

13.* In case of oblique axes we have

$$\begin{aligned}abc\,.\,\boldsymbol{S}=p\sqrt{}\{&\overline{ab\,.\,\omega|}^2+\overline{bc\,.\,\chi|}^2+\overline{ca\,.\,\psi|}^2\\&-2(ab\,.\,\omega|\,.\,bc\,.\,\chi|\,\underline{Y}+bc\,.\,\chi|\,.\,ca\,.\,\psi|\,.\,\underline{Z}+ca\,.\,\psi|\,.\,ab\,.\,\omega|\,\underline{X})\},\end{aligned} \tag{4}$$

since each is the six-fold volume of the tetraeder $O-ABC$, where $OA=a$, $OB=b$, $OC=c$, and $\underline{X}, \underline{Y}, \underline{Z}$ are cosines of the diedral $\measuredangle$s along X, Y, Z. Call the $\triangle$s BOC, COA, AOB the *planar intercepts* of the plane whose *axial* intercepts are a, b, c, and denote them doubled by A, B, C; then

$$abc\,.\,\mathbf{S} = p\,\sqrt{\{A^2 + B^2 + C^2 - 2(AB\,.\,\underline{Z} + BC\,.\,\underline{X} + CA\,.\,\underline{Y})\}}. \qquad (5)$$

In (4) put for a, b, c, p their values $-\frac{d}{l}, -\frac{d}{m}, -\frac{d}{n}, -Fd$; hence

$$F = \mathbf{S} : \sqrt{\{\overline{l\,.\,\chi|}^2 + \overline{m\,.\,\psi|}^2 + \overline{n\,.\,\omega|}^2 - 2(l\,.\,\chi|m\,.\,\psi|\,.\,\underline{Z} + m\,.\,\psi|\,.\,n\,.\,\omega|\underline{X} + n\,.\,\omega|l\,.\,\chi|\,\underline{Y})\}}. \qquad (6)$$

The analogy of this to the value of F in plane geometry becomes plain on writing each in determinant form:

$$F^2 = \begin{vmatrix} 1 & \underline{\omega} \\ \underline{\omega} & 1 \end{vmatrix} : \begin{vmatrix} 0 & l & m \\ l & 1 & \underline{\omega} \\ m & \underline{\omega} & 1 \end{vmatrix}; \quad F^2 = \begin{vmatrix} 1 & \underline{\omega} & \underline{\psi} \\ \underline{\omega} & 1 & \underline{\chi} \\ \underline{\psi} & \underline{\chi} & 1 \end{vmatrix} : \begin{vmatrix} 0 & l & m & n \\ l & 1 & \underline{\omega} & \underline{\psi} \\ m & \underline{\omega} & 1 & \underline{\chi} \\ n & \underline{\psi} & \underline{\chi} & 1 \end{vmatrix}. \qquad (7)$$

By reasoning like that in plane geometry it is now shown that the distance of (x, y, z) from the plane $x\underline{\alpha} + y\underline{\beta} + z\underline{\gamma} - p = 0$ is $x\underline{\alpha} + y\underline{\beta} + z\underline{\gamma} - p$, or (x, y, z) is distant $N(x, y, z)$ from $N(x, y, z) = 0$.

The whole body of reasoning as to *normal Eqs.* of *RLs.* may now be repeated as to *normal* Eqs. of *planes;* and as there the *Abridged Notation* issued in a system of *homogeneous triangular* (or trilinear) *Cds.*, so here it issues in a system of *homogeneous tetraedral* (or quadriplanar) Cds.; and just as the first could also be interpreted as *line*-Cds., so the second can also be interpreted as *plane*-Cds., a thought that cannot be developed here.

14. We have found the Eq. of a plane through three points and the six-fold volume of a tetraeder, given by its vertices, to be respectively

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0, \quad \text{and} \quad 6\mathbf{T} = \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} \mathbf{S}. \qquad (8)$$

These two Eqs. are really one, the first merely saying that the volume is 0 when the fourth point (x, y, z) is in the plane of the other three.

The same six-fold volume can be expressed as the product of the double base $P_1P_2P_3$ by the $\perp p$ from P on the plane of the base. This p is found by bringing the Eq. of the plane into the normal form, by multiplying the determinant by F, where

$$F = \boldsymbol{S} : \sqrt{\{\overline{x'\cdot\chi|}^2 + \overline{y'\cdot\psi|}^2 + \overline{z'\cdot\omega|}^2}$$
$$- 2(x'\cdot\chi|\cdot y'\cdot\psi|\underline{Z} + y'\cdot\psi|z'\cdot\omega|\underline{X} + z'\cdot\omega|x'\cdot\chi|\underline{Y})\},$$

where x', y', z' are co-factors of x, y, z in the determinant. Hence, the *radical* $\sqrt{\{\ \}}$ *is the double* **area** *of the* $\triangle$ $P_1P_2P_3$. We note that $x'\cdot\chi|$, $y'\cdot\psi|$, $z'\cdot\omega|$ are the projections of $P_1P_2P_3$ on the Cd. planes ‖ to X, Y, Z; i.e., *they are the* **Cds.** *of the* $\triangle$; also, the negative terms are their (so-called) inner products in sets of two; hence,

The squared area of a $\triangle$ is the sum of its squared Cds. plus twice the sum of their inner products, in sets of two; or

The squared area of a $\triangle$ is the inner squared sum of its Cds.

The Cds. of the tract P_1P_2 being the differences of the *Cds.* of its ends, by observing signs, the *square of the tract* may be expressed as above.

15. The *direction-∡s* α, β, γ of a $\perp$ on a plane are called the **Position-∡s** of the plane; clearly they are also the **diedral ∡s** of the plane with the Cd. *planes*, since the ∡ between two planes equals the ∡ between $\perp$s on them; they are the **complements** of the ∡s between the plane and the Cd. *axes.*

The **position-cosines** of a plane are $\boldsymbol{l} : \sqrt{\boldsymbol{l}^2 + \boldsymbol{m}^2 + \boldsymbol{n}^2}$, and two like ones.

15*. In case of oblique axes, the position-cosines are still $\boldsymbol{Fl}$, $\boldsymbol{Fm}$, $\boldsymbol{Fn}$, but the *position-∡s* no longer equal the *diedral ∡s.* But any diedral ∡, as between the plane and XY, equals the ∡ between the $\perp$ p on the plane and the $\perp$ p_z on XY. The direction-cosines of $\boldsymbol{p}$ are $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$, or Fl, Fm, Fn; those of p_z are 0, 0, $\boldsymbol{S} : \underline{\omega}$ (see Art. 4*); put these values in (C_2*): Fl, Fm, Fn for $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$, and 0, 0, $\boldsymbol{S} : \underline{\omega}$ for $\underline{\alpha}_1$, $\underline{\beta}_1$, $\underline{\gamma}_1$, and get

$$\underline{\widehat{pp_1}} = \underline{\widehat{pp_z}} = \{\omega|^2 Fn - \chi'' Fm - \psi'' Fl\} : \boldsymbol{S}\,.\,\omega|$$
$$= F\{n(1 - \underline{\omega}^2) - m(\underline{\chi} - \underline{\psi}\underline{\omega}) - l(\underline{\psi} - \underline{\omega}\underline{\chi})\} : \boldsymbol{S}\,.\,\underline{\omega}|.$$

The cosines of the other *diedral ∡s* are got by permuting symbols.

16. The ∡ between two planes π_1, π_2, whose Eqs. are

$$l_1x + m_1y + n_1z + d_1 = 0, \quad l_2x + m_2y + n_2z + d_2 = 0,$$

equals the ∡ between the $\perp$s on the planes; hence

$$\widehat{\pi_1\pi_2} = (l_1l_2 + m_1m_2 + n_1n_2) : \sqrt{l_1^2 + m_1^2 + n_1^2} \cdot \sqrt{l_2^2 + m_2^2 + n_2^2}.$$

The *slope* of a RL. to a plane is the *complement of the slope.* of the RL. to the $\perp$ on the plane; hence, if the Eq. of the plane be

$$lx + my + nz + d = 0,$$

and those of the RL. be

$$\frac{x - x_1}{\lambda} = \frac{y - y_1}{\mu} = \frac{z - z_1}{\nu},$$

$$\widehat{\pi L}| = (l\lambda + m\mu + n\nu) : \sqrt{l^2 + m^2 + n^2} \cdot \sqrt{\lambda^2 + \mu^2 + \nu^2}.$$

Hence, the *two planes* are $\perp$ when

$$\boldsymbol{l_1l_2 + m_1m_2 + n_1n_2 = 0}.$$

They are $\parallel$ when

$$l_1l_2 + m_1m_2 + n_1n_2 = \sqrt{l_1^2 + m_1^2 + n_1^2} \cdot \sqrt{l_2^2 + m_2^2 + n_2^2},$$

i.e., when

$$(l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 = 0.$$

This sum of squares is 0 only when each is 0; i.e., only when

$$l_1 : l_2 = m_1 : m_2 = n_1 : n_2.$$

This condition is, indeed, geometrically evident, since it declares only that the *position-cosines* of the two planes are the same.

The *plane and the RL.* are $\parallel$ when

$$l\lambda + m\mu + n\nu = 0.$$

They are $\perp$ when

$$l\lambda + m\mu + n\nu = \sqrt{l^2 + m^2 + n^2} \cdot \sqrt{\lambda^2 + \mu^2 + \nu^2};$$

i.e., when

$$l : \lambda = m : \mu = n : \nu.$$

Hence $lx + my + nz + d = 0$

and $\quad \dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n} \quad$ are $\perp$.

17. To find where a *line* meets a *surface*, replace *two* of the Cds. in the *Eq.* of the *surface* by their values in terms of the *third*, taken from the *Eqs.* of the *line;* thus is got *one* Eq. in *one* Cd., whose roots are Cds. of the points of meeting. The number of these roots cannot be *greater* than the number of common points, though it may be *less*, since to any value of z (say) may correspond several values of x and y. If the Eq. in the third Cd. reduces to $0 = 0$, i.e., is satisfied for *every* value of that Cd., *then, and only then*, the line has *all* of its points on the surface; i.e., **the line is on the surface.**

In the special case of the plane $lx + my + nz + d = 0$ and the RL. $y = tz + b,\quad x = uz + a$ we get

$$(lu + mt + n)z + (la + mb + d) = 0,$$

and this reduces to $0 = 0$, is satisfied for every z, only when

$$\boldsymbol{lu + mt + n = 0} \quad \text{and} \quad \boldsymbol{la + mb + d = 0,}$$

which Eqs. say *the RL. lies in the plane.*

18. The **common point** of three planes,

$$l_1x + m_1y + n_1z + d_1 = 0,$$
$$l_2x + m_2y + n_2z + d_2 = 0,$$
$$l_3x + m_3y + n_3z + d_3 = 0,$$

is found *by solving the three Eqs. as simultaneous;* the results are

$$x = -|d_1\ m_2\ n_3| : |l_1\ m_2\ n_3|,$$
$$y = -|l_1\ d_2\ n_3| : |l_1\ m_2\ n_3|,$$
$$z = -|l_1\ m_2\ d_3| : |l_1\ m_2\ n_3|.$$

Hence, if $|l_1\, m_2\, n_3| = 0$, the common point is at ∞, **the intersections are $\parallel$ RLs.**; if, besides, a numerator, as $|l_1\, m_2\, d_3|$, be 0, the common point becomes indefinite, the intersections fall together, **the three planes pass through the same RL.**

19. If four planes meet in a point, their four Eqs. are satisfied by the same triplet of values x, y, z; this can be *when, and only when,* $\boldsymbol{|l_1\, m_2\, n_3\, d_4| = 0.}$

Or we may reason otherwise, thus: If $\pi_1 = 0$, $\pi_2 = 0$, $\pi_3 = 0$ be three planes, then $\lambda_1\pi_1 + \lambda_2\pi_2 + \lambda\,\pi_3 = 0$ is a *plane through their common point;* for this Eq. represents a plane, being of first degree in x, y, z; and it is satisfied by that triplet of values x, y, z, that satisfies the three at once.

Four planes $\pi_1 = 0$, $\pi_2 = 0$, $\pi_3 = 0$, $\pi_4 = 0$ meet in a point when four multipliers λ_1, λ_2, λ_3, λ_4, can be found such that $\lambda_1\pi_1 + \lambda_2\pi_2 + \lambda_3\pi_3 + \lambda_4\pi_4 = 0$ identically; for the triplet of values x, y, z, which reduces any three of the π's to 0, must reduce the fourth π to 0 also.

20. If $N_1 = 0$, $N_2 = 0$, be normal Eqs. of two planes, then $N_1 - \lambda N_2 = 0$ is a *plane through their common RL.;* for any triplet x, y, z, satisfying the first two Eqs., satisfies the third. Also, $\lambda = N_1 : N_2$; i.e., λ is the *ratio* of the *distances* of any point of the third *plane* from the two *base-planes*, or λ is the *ratio* of the *sines* of the *slopes* of the third *plane* to the *base-planes*. Hence, $N_1 - N_2 = 0$ resp. $N_1 + N_2 = 0$ is the *inner* resp. *outer halver* of the ∡s between the *base-planes.*

21. To find the *direction-cosines* of a RL. *halving* the ∡ *between two RLs* directed by α, β, γ and α', β', γ', take two ∥s to the RLs., through the origin, and on each take a point distant 2 from the origin; the Cds. of the points will be $2\underline{\alpha}$, $2\underline{\beta}$, $2\underline{\gamma}$ and $2\underline{\alpha}'$, $2\underline{\beta}'$, $2\underline{\gamma}'$; the *mid-point* of the two will be $(\underline{\alpha} + \underline{\alpha}'$, $\underline{\beta} + \underline{\beta}'$, $\underline{\gamma} + \underline{\gamma}')$, and will be on the halver sought; hence

$$\frac{x}{\underline{\alpha} + \underline{\alpha}'} = \frac{y}{\underline{\beta} + \underline{\beta}'} = \frac{z}{\underline{\gamma} + \underline{\gamma}'}$$

is the *Eq. of the halver;* hence $\underline{\alpha} + \underline{\alpha}'$, $\underline{\beta} + \underline{\beta}'$, $\underline{\gamma} + \underline{\gamma}'$, each divided by $\sqrt{(\underline{\alpha} + \underline{\alpha}')^2 + (\underline{\beta} + \underline{\beta}')^2 + (\underline{\gamma} + \underline{\gamma}')^2}$ are the direction-cosines sought. The radical $\sqrt{\ }$ reduces to

$$\sqrt{2 + 2\underline{\phi}} = 2 \cdot \frac{\phi}{2}\Big|,$$

where ϕ is the ∡ *between the RLs.*

To find the direction-cosines of the outer halver, it suffices to change the signs of α', β', γ'; the radical $\sqrt{\ }$ then becomes

$$\sqrt{2-2\underline{\phi}} = 2\cdot\underline{\left(\frac{\phi}{2}\right)}.$$

22. A system of planes through a point may be called a *pencil of planes;* for a system of planes through a RL. no fitting name has yet been used in English; perhaps *cluster* would answer best to the German *Bueschel*, suggested by the phrase in Architecture, *clustered column.*

The *common RL.* of a *cluster* may be called its *axis.* Any two planes may be taken as base-planes of a cluster.

To find what plane of a cluster

$$l_1x+m_1y+n_1z+d_1-\lambda(l_2x+m_2y+n_2z+d_2)=0$$

is $\perp$ to $lx+my+nz+d=0$, we have at once, by Art. 16,

$$l(l_1-\lambda l_2)+m(m_1-\lambda m_2)+n(n_1-\lambda n_2)=0,$$

whence λ is to be found, and, on substitution, the Eq. of the plane is found to be

$$(ll_2+mm_2+nn_2)(l_1x+m_1y+n_1z+d_2)$$
$$-(ll_1+mm_1+nn_1)(l_2x+m_2y+n_2z+d_2)=0.$$

The position-cosines of this plane are proportional to

$$m(l_1m_2-l_2m_1)-n(n_1l_2-n_2l_1),$$

and two like expressions.

But l, m, n, are proportional to the position-cosines of the given plane, say $x\underline{\alpha}+y\underline{\beta}+z\underline{\gamma}-p=0$, and the parentheses are proportional to the direction-cosines of the axis, say $\frac{x-x'}{\underline{\alpha}'}=\frac{y-y'}{\underline{\beta}'}=\frac{z-z'}{\underline{\gamma}'}$; hence the above Eq. of the sought plane may also be written

$$(x-x')(\underline{\beta}\underline{\gamma}'-\underline{\beta}'\underline{\gamma})+(y-y')(\underline{\gamma}\underline{\alpha}'-\underline{\gamma}'\underline{\alpha})$$
$$+(z-z')(\underline{\alpha}\underline{\beta}'-\underline{\alpha}'\underline{\beta})=0.$$

23. To find the plane through either of two RLs. ∥ to the other, regard the RLs. as the axes of the clusters $\Pi_1-\lambda\Pi_2=0$, $\Pi_3-\kappa\Pi_4=0$. The two ∥ planes of these clusters are the planes sought.

By Art. 16

$$\frac{l_1-\lambda l_2}{l_3-\kappa l_4}=\frac{m_1-\lambda m_2}{m_3-\kappa m_4}=\frac{n_1-\lambda n_2}{n_3-\kappa n_4}=r \text{ (say)}.$$

Clear each of the three Eqs. and solve for λ; so we get

$$\lambda=|l_1\, m_3\, n_4|:|l_2\, m_3\, n_4|,$$

whence $\quad \Pi_1|l_2\, m_3\, n_4|-\Pi_2|l_1\, m_3\, n_4|=0,$

and advancing the subscripts by 2,

$$\Pi_3|l_4\, m_1\, n_2|-\Pi_4|l_3\, m_1\, n_2|=0,$$

are the planes sought.

Like results are reached by this reflection: The common ⊥ to the two RLs., say

$$\frac{x-x'}{\underline{\alpha}'}=\frac{y-y'}{\underline{\beta}'}=\frac{z-z'}{\underline{\gamma}'} \text{ and } \frac{x-x''}{\underline{\alpha}''}=\frac{y-y''}{\underline{\beta}''}=\frac{z-z''}{\underline{\gamma}''},$$

is clearly ⊥ to the plane having the direction of both, ∥ to both or through either ∥ to the other; the direction-cosines of this ⊥ have already been found proportional to $\underline{\beta}'\underline{\gamma}''-\underline{\beta}''\underline{\gamma}'$, etc.; hence the planes are

$$(x-x')(\underline{\beta}'\underline{\gamma}''-\underline{\beta}''\underline{\gamma}')+(y-y')(\underline{\gamma}'\underline{\alpha}''-\underline{\gamma}''\underline{\alpha}')$$
$$+(z-z')(\underline{\alpha}'\underline{\beta}''-\underline{\alpha}''\underline{\beta}')=0,$$

and $\quad (x-x'')(\underline{\beta}'\underline{\gamma}''-\underline{\beta}''\underline{\gamma}')+\cdots$

The *distance between these planes* from any point, as (x', y', z'), of the first to the second, is plainly the (shortest) **distance between the RLs.**; the same is got by reducing the Eqs. to the normal form, dividing by the second root of the sum of the squared coefficients of x, y, z, and then putting x', y', z' for x, y, z in the second Eq.; now that sum is the squared sine of the ∡ between the RLs. (Exercise, Art. 6); hence, calling the distance d and the ∡ ϕ, we have

$$d\cdot\phi| = (x'-x'')(\underline{\beta}'\underline{\gamma}''-\underline{\beta}''\underline{\gamma}') + (y'-y'')(\underline{\gamma}'\underline{\alpha}''-\underline{\gamma}''\underline{\alpha}') + (z'-z'')(\underline{\alpha}'\underline{\beta}''-\underline{\alpha}''\underline{\beta}').$$

This expression, called by Cayley the *moment* of the two RLs., may be written as the difference of two determinants, thus:

$$d\cdot\phi| = \begin{vmatrix} x' & \underline{\alpha}' & \underline{\alpha}'' \\ y' & \underline{\beta}' & \underline{\beta}'' \\ z' & \underline{\gamma}' & \underline{\gamma}'' \end{vmatrix} - \begin{vmatrix} x'' & \underline{\alpha}' & \underline{\alpha}'' \\ y'' & \underline{\beta}' & \underline{\beta}'' \\ z'' & \underline{\gamma}' & \underline{\gamma}'' \end{vmatrix},$$

which clearly

$$= -\{\underline{\alpha}'(\underline{\beta}''z''-\underline{\gamma}''y'') + \underline{\beta}'(\underline{\gamma}''x''-\underline{\alpha}''z'') + \underline{\gamma}'(\underline{\alpha}''y''-\underline{\beta}''x'') + \underline{\alpha}''(\underline{\beta}'z'-\underline{\gamma}'y') + \underline{\beta}''(\underline{\gamma}'x'-\underline{\alpha}'z') + \underline{\gamma}''(\underline{\alpha}'y'-\underline{\beta}'x')\}.$$

Now the Greek letters are the direction-cosines of the two RLs., and are but special values of λ', μ', ν' and λ'', μ'', ν''; while the parentheses are what, consistently with Art. 11, must be denoted by Λ'', M'', N'' and Λ', M', N'; hence, disregarding sign,

$$d\cdot\phi| = \lambda'\Lambda''+\mu'M''+\nu'N''+\lambda''\Lambda'+\mu''M'+\nu''N',$$

which expresses *the* **moment** *of two RLs. through their Cds.*

24. To find the *volume of a tetraeder* fixed by *four planes*, $\Pi_1=0$, etc., it suffices to repeat, step by step, the reasoning in Plane Geometry as to the area of a △ fixed by three RLs.

The result is quite of like form:

$$6\,T = |l_1 m_2 n_3 d_4|^3 : |l_2 m_3 n_4|\cdot|l_1 m_3 n_4|\cdot|l_1 m_2 n_4|\cdot|l_1 m_2 n_3|.$$

25. It is easy now to interpret $\Lambda = \underline{\beta}z'-\underline{\gamma}y'$, M and N. Let $\frac{x-x'}{\underline{\alpha}} = \frac{y-y'}{\underline{\beta}} = \frac{z-z'}{\underline{\gamma}}$ be the RL., then $\frac{x}{\underline{\alpha}'} = \frac{y}{\underline{\beta}'} = \frac{z}{\underline{\gamma}'} = \delta$ is the Eq. of the RL. through the origin and (x', y', z'); hence $x'=\underline{\alpha}'\delta$, $y'=\underline{\beta}'\delta$, $z'=\underline{\gamma}'\delta$, and $\Lambda=\delta(\underline{\beta}\underline{\gamma}'-\underline{\gamma}\underline{\beta}')$, $M=\delta(\underline{\gamma}\underline{\alpha}'-\underline{\gamma}'\underline{\alpha})$, $N=\delta(\underline{\alpha}\underline{\beta}'-\underline{\alpha}'\underline{\beta})$. The multipliers of δ are the position-cosines of the plane through the two RLs.; i.e., of the plane through the given RL. and the origin; or, they are the direction-cosines of the RL. through the origin $\perp$ to this plane, and hence Λ, M, N are the Cds. of a point on this RL. distant δ from the origin; i.e., the *Cds. of a point on this RL.* as far from the origin *as* (x', y', z') *is.*

EXERCISES.

1. The tract $\overline{P_1P_2}$ is cut at P' in ratio $n_1 : n_2$, $\overline{P'P_3}$ is cut at P'' in ratio $\overline{n_1 + n_2} : n_3$, $\overline{P''P_4}$ at P''' in ratio $\overline{n_1 + n_2 + n_3} : n_4$; find the Cds. of P''', the *centre of proportional distances* of P_1, P_2, P_3, P_4.

2. Find distance of (x, y, z) from a RL. through O directed by α, β, γ.

3. If $\pi_1 = 0$, $\pi_2 = 0$, $\pi_3 = 0$, $\pi_4 = 0$ do *not* meet in a point, the Eq. of any plane is of the form $\lambda_1\pi_1 + \lambda_2\pi_2 + \lambda_3\pi_3 + \lambda_4\pi_4 = 0$.

Show that the π's, regarded as *Cds. of a point*, are *proportional* to fixed multiples of its *distances* from the planes; the λ's, regarded as *Cds. of a plane*, to fixed multiples of its *distances* from the points $\lambda_1 = 0$, etc.

4. Find Eq. of: a plane through two ∥ RLs., a RL. through two points.

5. Show that the Eq. of a *sphere* of radius r, centre (x_1, y_1, z_1) is

$$\overline{x - x_1}^2 + \overline{y - y_1}^2 + \overline{z - z_1}^2 = r^2.$$

6. What, then, are the Eqs. of a circle in space?

7. Find the centres of the inscribed and circumscribed circles of a △ whose vertices are on the rectang. axes.

8. Show that the three *median* planes of a trieder (through the edges halving the counter-*sides* or *face-*$\measuredangle$*s*) meet in a RL.

9. Three planes through the three edges of a trieder meet in a RL.; show that the compound ratio of the sines of the segments into which they cut the counter-sides is 1; and conversely.

10. Any plane through the vertex of a trieder cuts the sides into segments the compound ratio of whose sines is -1; and conversely.

11. The 6 planes of intersection of 4 spheres meet in a point.

12. Three positive rectang. axes pierce a sphere about O at X, Y, Z; X' is the pole of the circle through X, Y, Z; the point X is carried up to X' along the great circle arc XX'; find how Y and Z move, and the formulæ of transformation from axes OX, OY, OZ to OX', OY', OZ'.

CHAPTER II.

SURFACES OF SECOND DEGREE.

(Quadrics or Conicoids.)

26. The general Eq. of second degree has the form

$$kx^2+2hxy+jy^2+2gzx+2fyz+iz^2+2lx+2my+2nz+d=0.$$

It shall be referred to as $F(x, y, z; x, y, z)=0$, or $\boldsymbol{Q}=\mathbf{0}$. Before discussing it, certain general notions shall be premised.

A surface may be thought as the *locus* of a *point;* it may also be thought as the *locus* of a *line*, or traced by a *moving line*.

Suppose two surfaces $F(x, y, z; p)=0$, $\phi(x, y, z; p)=0$, contain the same parameter $\boldsymbol{p}$; for any special value of p they fix a single line as their intersection, while various values of p yield various such lines; by eliminating p between the Eqs. we get a relation holding between x, y, z for all points on all such lines; i.e., we get the Eq. of the *locus* of the *line*, the surface traced by it *moving*. If F and ϕ contain two parameters p and p', they must be bound together by some Eq., as $f(p, p')=0$; the number of such Eqs. of condition is, in general, *one less* than the number of parameters.

The moving line is called the **generatrix** (in any one position it is an **element**) of the surface. The motion of the generatrix is commonly defined as *gliding along* fixed lines called **directrices.** Since the Eqs. of generatrix and a directrix hold for the same triplet x, y, z, by eliminating these from the four Eqs. is got one Eq. of condition between the parameters for *each directrix;* hence, when the Eq. of generatrix contains $\boldsymbol{n}$ parameters, it must glide on $\boldsymbol{n-1}$ directrices.

When the generatrix is a RL., the surface is called **ruled.** The Eq. of the RL. contains *four* parameters, hence *three* directrices define its motion. *Ruled* surfaces that can be unwrapped upon a plane are called *developable*; the generatrix of such a surface always touches a fixed curve called *cuspidal edge*. Other ruled surfaces are called warped or *twisted*.

27. *A* **cylindric** *surface is the path of a RL.* **pushed.**

Be $y = tz + b$, $x = uz + a$ the RL.; then, since the direction of the RL. changes not, t and u are *constant;* let $\phi(a, b) = 0$ be the Eq. between the parameters a and b, yielded by the Eq. of the directrix; on replacing a and b by their values results

$$\phi(x - uz, y - tz) = 0$$

as *Eq. of the cylinder.*

Or, be $lx + my + nz + d = p$, $l'x + m'y + n'z + d' = p'$ the Eqs. of the RL.; letting only p and p' vary, we keep each plane ∥ to itself, and hence all the intersections ∥; if $\phi(p, p') = 0$ be the Eq. between the parameters, the Eq. of the cylinder is

$$\phi(lx + my + nz + d,\ l'x + m'y + n'z + d') = 0.$$

Hence *any Eq. of a cylinder is an Eq. between two functions of first degree in x, y, z;* and the *converse* is clear.

28. *A* **conic surface** *is the path of a RL.* **turned** (about a point).

Be $\frac{x - x_1}{z - z_1} = p$, $\frac{y - y_1}{z - z_1} = q$ the RL., (x_1, y_1, z_1) being the fixed point about which it turns; then, just as above,

$$\phi\left(\frac{x - x_1}{z - z_1}, \frac{y - y_1}{z - z_1}\right) = 0$$

is the *Eq. of the conic surface* (or *cone*). We note the Eq. is *homogeneous* in *Cd. differences;* the converse is clear, that *every Eq. homogeneous in Cd. differences pictures a cone.*

If the fixed point, or vertex of the cone, be the origin, the Eq. is *homogeneous* in x, y, z; and conversely.

Both cylinder and cone are *developable;* the *cuspidal edge* of the cone is reduced to a point, the vertex, while the cylinder is but a cone with its *vertex at* ∞.

29. *A* **surface of revolution** *is the path of a line* **revolved** *about a fixed RL., the* **axis,** to which it is supposed rigidly attached. Or, it is the *path of a circle* whose (varying) diameter is always halved by a fixed RL., the *axis*, at *right angles.*

The *generating circle* in any position is called a **parallel,** the *revolving line* in any position, a **meridian,** of the surface.

If $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ be the *axis*, then $lx+my+nz=p$ is a *plane* ⊥ *to it*, and $\overline{x-x_1}^2+\overline{y-y_1}^2+\overline{z-z_1}^2=r^2$ is a *sphere* about x_1, y_1, z_1; any *parallel* is the intersection of two of these surfaces; hence, exactly as before,

$$\phi(\overline{x-x_1}^2+\overline{y-y_1}^2+\overline{z-z_1}^2,\ lx+my+nz)=0$$

is the general *Eq. of a surface of revolution.*

30. Returning to the Eq. $Q=0$, we note it may be written

$$(kx+hy+gz+l)x+(hx+jy+fz+m)y$$
$$+(gx+fy+iz+n)z+(lx+my+nz+d)=0. \quad (1)$$

The parentheses may be called, in order, Q_x, Q_y, Q_z, Q_1.

Again, we note there are ten coefficients; but, by dividing by any one, the number is reduced to nine; these may be determined by nine independent Eqs.; hence, *nine simple conditions* are *needed* and *enough* to *determine* a *quadric.*

To pass to ∥ axes through a new origin, x', y', z', put $x+x'$, $y+y'$, $z+z'$ for x, y, z; then, by reasoning quite like that in Plane Geometry, the result is seen to be

$$kx^2+2hxy+jy^2+2gzx+2fyz+iz^2$$
$$+2Q_x'\cdot x+2Q_y'\cdot y+2Q_z'z+Q'=0, \quad (2)$$

where $Q_x'=kx'+hy'+gz'+l$, and so for the others.

We note that the coefficients of terms of second degree change not.

When, and *only when*, all the Q's vanish, does the Eq. become homogeneous in x, y, z; but then it represents a cone through the new origin x', y', z'. Now $Q' = Q'_x \cdot x' + Q'_y \cdot y' + Q'_z \cdot z' + Q'_1$; hence, for Q'_x, Q'_y, Q'_z, Q' each to vanish is the same as for Q'_x, Q'_y, Q'_z, Q'_1 each to vanish; and these four vanish for the same triplet x', y', z' *when*, and *only when*,

$$\Delta = \begin{vmatrix} k & h & g & l \\ h & j & f & m \\ g & f & i & n \\ l & m & n & d \end{vmatrix} = 0.$$

Hence, **$Q = 0$ represents a cone** when, and only when, **$\Delta = 0$.** This Δ may be called the **discriminant** of $Q = 0$.

If $\Delta = 0$, then $kx^2 + 2hxy + jy^2 + 2gzx + 2fyz + iz^2 = 0$, and this breaks up into *two linear factors* in x, y, z; i.e., the cone breaks up into *two planes*, when, and only when,

$$D = \begin{vmatrix} k & h & g \\ h & j & f \\ g & f & i \end{vmatrix} = 0,$$

as was proved in Plane Geometry. Hence $Q = 0$ **represents two planes** when, and only when, **$\Delta = 0$, $D = 0$.**

31. If $Q'_x = 0$, $Q'_y = 0$, $Q'_z = 0$, but $Q'(\text{or } Q'_1) \lesseqgtr 0$, then if any *triplet* (x, y, z) satisfies the new Eq., so does the *counter-triplet* $(-x, -y, -z)$, since the Cds. appear *only in pairs;* i.e, if *any point* be on the surface, so is its *counter-point* as to the new origin; i.e., *the new origin halves every chord through it*, and is the **centre.**

The *Cds. of this centre*, or x', y', z', are found from the three Eqs., $Q'_x = 0$, $Q'_y = 0$, $Q'_z = 0$ to be $L : D$, $M : D$, $N : D$. Hence, if **$D \lesseqgtr 0$,** the centre is *infinity*, and the surface is **centric**; if **$D = 0$,** the centre is *at* ∞, the surface is called **non-centric.**

One or more of the numerators L, M, N may vanish along with D; the centre is then *indeterminate.*

In case Q' alone $= 0$, the *origin* (x', y', z') is *on the surface.* Call the sum of the six terms of second degree S; then the Eq. becomes

$$S + 2(xQ'_x + yQ'_y + zQ'_z) = 0 \qquad (3)$$

Draw through the origin any RL. $x : \underline{\alpha} = y : \underline{\beta} = z : \underline{\gamma} = \rho.$ To find where it meets the surface, replace x, y, z in (2); hence

$$\Sigma\rho^2 + 2(\underline{\alpha}Q'_x + \underline{\beta}Q'_y + \underline{\gamma}Q'_z)\rho = 0.* \qquad (4)$$

One root ρ_1 of this Eq. is always 0; the *other*, ρ_2, is also 0 *when*, and *only when*,

$$\underline{\alpha}Q'_x + \underline{\beta}Q'_y + \underline{\gamma}Q'_z = 0,$$

or when $\quad x \cdot Q'_x + y \cdot Q'_y + z \cdot Q'_z = 0; \qquad (5)$

i.e., *the RL. meets the surface* only at the origin, and then *in two consecutive points*, when, and only when, it *lies in the* **plane** whose Eq. is (5).

A RL. meeting a surface in **two consecutive points** is **tangent** to the surface; the **plane** *containing all tangents* to a surface at a point is **tangent** to the surface at that point. Hence (5) is the *plane tangent* at the *origin*.

To find the Eq. of this plane tangent at (x', y', z') in terms of the *old* Cds., replace x, y, z by $x - x', y - y', z - z'$; hence,

$$(x - x')Q'_x + (y - y')Q'_y + (z - z')Q'_z = 0. \qquad (6)$$

Add $\quad x'Q'_x + y'Q'_y + z'Q'_z + Q'_1 = 0,$

since (x', y', z') is on the surface; hence,

$$\begin{aligned} xQ'_x + yQ'_y + zQ'_z + Q'_1 &= 0 \\ &= x'Q_x + y'Q_y + z'Q_z + Q_1; \end{aligned} \qquad (7)$$

or, $\quad F(x, y, z; x', y', z') = 0 = F(x', y', z'; x, y, z), \quad (7*)$

is the **Eq. of the plane tangent to** $F(x, y, z; x, y, z) = 0$ at (x', y', z').

32. The *meaning* of the fact that this Eq. is **like-formed** as to x, y, z and x', y', z' is quite like the meaning of the like fact in Plane Geometry, and is developed in like way. In fact, if (x', y', z') be *not* on the surface, the *Eq. still represents a plane*,

* Σ is what S becomes on putting $\underline{\alpha}, \underline{\beta}, \underline{\gamma}$ for x, y, z.

being of first degree; also, if a plane through (x', y', z') touches $Q=0$ at (x_1, y_1, z_1), its Eq. is $F(x_1, y_1, z_1; x, y, z)=0$, and hence $F(x_1, y_1, z_1; x', y', z')=0$; but this Eq. also says that (x_1, y_1, z_1) is on $F(x, y, z; x', y', z')=0$: hence this last Eq. is that of a **plane through all points of tangency** of planes through (x', y', z'). Such a plane is called the **polar** (plane) of (x', y', z') *as to* $Q=0$.

33. Hence it appears that all *tangent-planes*, and hence all *tangent-lines*, through a point, touch a quadric (surface of second degree) along a *plane-section* of that surface. But a *plane-section* is clearly a *conic;* for the section made by the XY-plane is found, by putting $z=0$, to be the conic

$$kx^2+2hxy+jy^2+2lx+2my+d=0,$$

and *any* plane may be taken as XY-plane without changing the degree of the Eq. or its general form. Hence *all tangents through a point*, or the **tangent-cone** through a point, *touch the quadric along a* **conic**.

We may note in passing that $\|$ *plane-sections of a quadric are similar conics.* For they are got by giving different constant values to z, as c, c', etc; but these do not affect the *first three terms* of the conic, on whose coefficients alone, k, h, j, the *shape* of the conic depends.

34. The *whole theory of poles and polars*, since it depends solely on the symmetry of the Eq. of the tangent-RL. resp.-plane as to the current Cds. and Cds. of the pole, may now be repeated from Plane Geometry.

Poles lying each on the polar of the other are *conjugate.*

Planes each through the pole of the other are *conjugate.*

Tangent-planes along a conic on a quadric go through a point.

Poles of planes through a point lie on a plane.

As a pole moves about on a plane, its polar-plane turns about a point; as a plane turns about a point, its pole moves about in a plane. A tract from a pole to its polar-plane is cut *harmonically* by the quadric (*referee*).

35. If $(L:D,\ M:D,\ N:D)$ be taken as pole, then Q'_x, Q'_y, Q'_z vanish, and the polar is $0x+0y+0z+Q'_1=0$; i.e., the *polar* is the *plane* at ∞. Hence, the *polars of all points at* ∞ pass through this point, the *centre*. All RLs. through the same point at ∞ are ‖; this point being the *outer mid-point* of all intercepts (chords) of the quadric on these RLs., their *inner mid-points* lie on the *polar* of the point at ∞; this *central* plane accordingly *halves all* ‖ *chords* through its pole (at ∞); i.e., halves all its *conjugate chords*. Hence it is called a **diametral** *plane*. Among all these chords is one *central* one, which is therefore a **diameter** *conjugate* to the **diametral** *plane*. The section of the *diametral* plane, being a conic, itself has an ∞ of pairs of *conjugate diameters;* any one of these forms with the common conjugate diameter a *triplet* of *conjugate diameters;* the three planes fixed by the triplet of conjugate diameters form a *triplet* of *conjugate diametral planes*. Each *plane* halves all chords ‖ to the *intersection* of the other two.

The poles of a system of ‖ planes lie on the diameter conjugate to the planes. The *central distances* of a pole and its polar, measured on the diameter through the pole (conjugate to the polar), have for their *geometric mean* the half of that conjugate diameter. Tangent planes at the *ends* of a *diameter* are ‖ to its conjugate diametral plane.

36. The notion of diametral plane may be got otherwise, thus:

Be $$\frac{x-x'}{\underline{\alpha}}=\frac{y-y'}{\underline{\beta}}=\frac{z-z'}{\underline{\gamma}}=\rho$$

or $$x=x'+\underline{\alpha}\rho,\quad y=y'+\underline{\beta}\rho,\quad z=z'+\underline{\gamma}\rho$$

a RL.; combining with the Eq. of the quadric, we get

$$\Sigma\rho^2+2\mathrm{T}\rho+\Upsilon=0; \qquad (1)$$

when Σ has its former meaning,

$$\mathrm{T} = Q'_x \cdot \underline{a} + Q'_y \cdot \underline{\beta} + Q'_z \cdot \underline{\gamma},$$

and, lastly,

$$\Upsilon = Q'_x \cdot x' + Q'_y \cdot y' + Q'_z \cdot z' + 2\,Q'_1 + d.$$

Eq. (1) has two roots, ρ_1, ρ_2; i.e., *every RL. meets a quadric in two, and only two, points.* These roots are *counter* when, and only when, $\mathrm{T}=0$; i.e., for $\underline{a}$, $\underline{\beta}$, $\underline{\gamma}$ held constant (for ‖ RLs.), the intercept or chord of the quadric is *halved* by (x', y', z') *so long as the point* (x', y', z') *lies in the plane,* $T = 0$; this latter *is* the Eq. of a plane, being linear in x', y', z'.

From the Eq. of this *diametral plane,*

$$\mathrm{T} = Q'_x \cdot \underline{a} + Q'_y \cdot \underline{\beta} + Q'_z \cdot \underline{\gamma} = 0,$$

it is seen that *all diametral planes* form a *pencil* through the *intersection* of $Q'_x = 0$, $Q'_y = 0$, $Q'_z = 0$; i.e., the *centre.*

37. To find the *mutual slope* of *conjugate plane and chords,* put σ, τ, υ, ϙ (read *koppa*) for $k\underline{a} + h\underline{\beta} + g\underline{\gamma}$, $h\underline{a} + j\underline{\beta} + f\underline{\gamma}$, $g\underline{a} + f\underline{\beta} + i\underline{\gamma}$, $l\underline{a} + m\underline{\beta} + n\underline{\gamma}$; then is

$$\mathrm{T} = \sigma x' + \tau y' + \upsilon z' + ϙ = 0,$$

and the position-cosines $\underline{a}'$, $\underline{\beta}'$, $\underline{\gamma}'$ of this diametral plane are $\underline{a}' = \sigma : R$, $\underline{\beta}' = \tau : R$, $\underline{\gamma}' = \upsilon : R$, where $R^2 = \sigma^2 + \tau^2 + \upsilon^2$.

Hence, if ϕ be the slope in question,

$$\phi| = \underline{a}\underline{a}' + \underline{\beta}\underline{\beta}' + \underline{\gamma}\underline{\gamma}' = (\sigma\underline{a} + \tau\underline{\beta} + \upsilon\underline{\gamma}) : R = \Sigma : R.$$

The important question arises: *Are conjugates ever perpendicular? If so,* $\phi = 90°$, $\phi| = 1$, $\Sigma = R$, $\underline{a} = \underline{a}'$, $\underline{\beta} = \underline{\beta}'$, $\underline{\gamma} = \underline{\gamma}'$; whence,

$$\underline{a} = \sigma : R,\ \ \underline{\beta} = \tau : R,\ \ \underline{\gamma} = \upsilon : R; \qquad (1)$$

or,

$$(\sigma\underline{a} + \tau\underline{\beta} + \upsilon\underline{\gamma})\underline{a} = \sigma,$$

$$(\sigma\underline{a} + \tau\underline{\beta} + \upsilon\underline{\gamma})\underline{\beta} = \tau,$$

$$(\sigma\underline{a} + \tau\underline{\beta} + \upsilon\underline{\gamma})\underline{\gamma} = \upsilon.$$

Here then are the three Eqs. to determine the α, β, γ of a *chord* $\perp$ *to its conjugate diametral plane;* there is a fourth Eq. connecting them, $\underline{\alpha}^2+\underline{\beta}^2+\underline{\gamma}^2=1$, but this imposes *no new* fourth condition, since it can be got from the three by multiplying by $\underline{\alpha}, \underline{\beta}, \underline{\gamma}$ in turn, adding and cancelling. Actually to find α, β, γ from these four Eqs. would be very tedious; it is better to determine R or Σ, and thence α, β, γ. On replacing σ, τ, v in (1) by their values there result

$$(k-R)\underline{\alpha}+h\underline{\beta}+g\underline{\gamma}=0,$$
$$h\underline{\alpha}+(j-R)\underline{\beta}+f\underline{\gamma}=0,$$
$$g\underline{\alpha}+f\underline{\beta}+(i-R)\underline{\gamma}=0.$$

Divide in turn by hg, fh, gf; add in turn $\underline{\alpha}:f, \underline{\beta}:g, \underline{\gamma}:h$; put A, B, C for

$$k-\frac{hg}{f},\quad j-\frac{fh}{g},\quad i-\frac{gf}{h};$$

also put U for $\underline{\alpha}:f+\underline{\beta}:g+\underline{\gamma}:h$;

hence result $U=\underline{\alpha}(R-A):hg,$

$$U=\underline{\beta}(R-B):fh,$$
$$U=\underline{\gamma}(R-C):gf;$$

or, $\underline{\alpha}=hgU:(R-A),$

$$\underline{\beta}=fhU:(R-B),$$
$$\underline{\gamma}=gfU:(R-C).$$

Squaring, adding, and re-membering $\underline{\alpha}^2+\underline{\beta}^2+\underline{\gamma}^2=1$, we get

$$1:U=\sqrt{\{h^2g^2:\overline{R-A}^2+f^2h^2:\overline{R-B}^2+g^2f^2:\overline{R-C}^2\}};\quad (2)$$

and on dividing in turn by f, g, h, and adding, we get

$$U=\frac{hg}{f}\cdot\frac{U}{R-A}+\frac{fh}{g}\cdot\frac{U}{R-B}+\frac{gf}{h}\cdot\frac{U}{R-C};$$

whence $1 = \frac{hg}{f} \cdot \frac{1}{R-A} + \frac{fh}{g} \cdot \frac{1}{R-B} + \frac{gf}{h} \cdot \frac{1}{R-C},$

or $U = 0.$ (3)

On multiplying by $(R-A)(R-B)(R-C)$, and by $\kappa = 1 : hgf$, results

$$(R-A)(R-B)(R-C)\left\{\kappa - \frac{1}{f^2(R-A)} - \frac{1}{g^2(R-B)} - \frac{1}{h^2(R-C)}\right\} = 0.$$

This Eq. is of third degree in R, hence has at least **one** *real root*. To decide about the other roots, suppose $\kappa +$ and $A < B < C$; then, calling the left side of the Eq. E, we see that

for $R = -\infty$, E is $-$; for $R = A$, E is $-$;

for $R = B$, E is $+$; for $R = C$, E is $-$;

for $R = +\infty$, E is $+$.

In case κ is $-$, a change of sign in E takes place between $R = -\infty$ and $R = A$, instead of between $R = C$ and $R = \infty$. In *any* case *E changes sign thrice* as R passes through real values from $-\infty$ to $+\infty$; i.e., *in any case the Eq.* $E = 0$ *has three real roots:* R_1, R_2, R_3.

These real values of R give three real triplets of values of α, β, γ; hence, there are, *in general*, **three, and only three, diametral planes ⊥ to their conjugate chords.** They are called **chief** (or principal) **planes** of the quadric.

38. Each of these roots R_1, R_2, R_3 must satisfy Eq. (3); hence result three Eqs.; take second from first, and multiply by hgf; hence,

$$\overline{R_2 - R_1}\left\{\frac{h^2g^2}{(R_1-A)(R_2-A)} + \frac{f^2h^2}{(R_1-B)(R_2-B)} + \frac{g^2f^2}{(R_1-C)(R_2-C)}\right\} = 0.$$

Since $\overline{R_2 - R_1}$ cannot $=0$, the parenthesis $\{\ \}$ must $=0$; and two like Eqs. are got by permuting the indices.

Now
$$\underline{\alpha}_1 = hgU_1 : (R_1 - A),$$
$$\underline{\beta}_1 = fhU_1 : (R_1 - B),$$
$$\underline{\gamma}_1 = gfU_1 : (R_1 - C);$$

and so for the indices 2 and 3. Calling the three directions of the diametral planes, or what is tantamount, of their conjugate chords, d_1, d_2, d_3, and dividing by the U's, since they can none of them $=0$, we get $\underline{\widehat{d_1 d_2}} = 0$, $\underline{\widehat{d_2 d_3}} = 0$, $\underline{\widehat{d_3 d_1}} = 0$; i.e., **the three chief planes of a quadric are $\perp$ to each other.**

39. In the special case $U = 0$, follow also $R = A = B = C$; hence,
$$k = A + \frac{hg}{f}, \qquad \sigma = A\underline{\alpha} + hgU, \quad R = A + hgfU^2,$$
whence $U - fU^2\underline{\alpha} = 0$, $U - gU^2\underline{\beta} = 0$, $U - hU^2\underline{\gamma} = 0$.

Hence $\underline{\alpha} = 1 : fU$, $\underline{\beta} = 1 : gU$, $\underline{\gamma} = 1 : hU$, or $U = 0$.

Squaring and adding the first three Eqs., we get
$$U = \sqrt{\{h^2g^2 + f^2h^2 + g^2f^2\}} : hgf.$$

This is the same value of U as is given by Eq. (2) when $A = B = C$; but *besides* the triplet $\underline{\alpha}$, $\underline{\beta}$, γ thus got, the problem is solved by any other triplet $\underline{\alpha}'$, $\underline{\beta}'$, $\underline{\gamma}'$ that makes $U = 0$ or makes $\frac{\underline{\alpha}'}{f} + \frac{\underline{\beta}'}{g} + \frac{\underline{\gamma}'}{h} = 0$. This Eq. is satisfied in an ∞ of ways and whenever $U(\underline{\alpha}\underline{\alpha}' + \underline{\beta}\underline{\beta}' + \underline{\gamma}\underline{\gamma}') = 0$, since $1 : f = U\underline{\alpha}$, $1 : g = U\underline{\beta}$, $1 : h = U\underline{\gamma}$.

Now U is here *not* $= 0$; hence
$$\underline{\alpha}\underline{\alpha}' + \underline{\beta}\underline{\beta}' + \underline{\gamma}\underline{\gamma}' = 0;$$

i.e., *every direction* $\perp$ to the *direction* (α, β, γ) is a *chief direction.*

It is easy to prove analytically, but it is also clear geometrically, that this *special* case is the case of *surfaces of revolution;* the direction (α, β, γ) is that of the axis.

If $k=j=i$ and $h=g=f$, the surface is a sphere; every triplet (α, β, γ) fulfils the conditions, every direction is a chief one.

40. Returning to the Eq. of a diametral plane $T=0$, on putting for the Q's their values, it becomes clear that a triplet of conjugate *diametrals* are ‖ to the *Cd. planes* when, and only when, $h=0$, $g=0$, $f=0$, the reasoning being quite like the corresponding in Plane Geometry. Hence, by choosing as Cd. planes three planes ‖ to a set of conjugates, we make the terms in xy, yz, zx vanish. This can always be done.

Again, by choosing the centre as origin, we make the terms in x, y, z vanish. This can be done only when the centre is in finity. But when the centre is at ∞, the origin can be taken on the surface, making the absolute vanish, and also the term in z^2, a diameter being taken as Z-axis. Hence the forms reduce to

$$kx^2+jy^2+iz^2=d \quad \text{and} \quad kx^2+jy^2=2nz.$$

When d and n are not $=0$, the varieties of these are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\pm 1, \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{a^2}=\pm 1,$$

and $$\frac{x^2}{a}\pm\frac{y^2}{b}=2z.$$

The first is an *ellipsoid:* real resp. imaginary.

The second is an *hyperboloid:* single resp. double.

The third is a *paraboloid:* elliptic resp. hyperbolic.

When $d=0$ or $n=0$, the Eq. becomes homogeneous, and so represents a *cone*, and, in case another coefficient vanishes, still more specially a *cylinder*. These *limiting* cases the student himself can readily trace out.

The *rectangular* conjugate diametrals recommend themselves as Cd. planes; to indicate that *oblique* conjugates are chosen, it suffices, as in Plane Geometry, to *accent* the constants a, b, c.

41. Returning to the Eq. $\Sigma\rho^2 + 2\mathrm{T}\rho + \Upsilon = 0$, which fixes the distances ρ_1, ρ_2 from (x', y', z') in the direction (α, β, γ) to the quadric $Q = 0$, we note that Σ depends only on the direction (α, β, γ), and Υ only on the point (x', y', z'); hence, taking *two* points P' and P'', and *one* direction, we find the quotient of the distance-products $\rho_1'\rho_2' : \rho_1''\rho_2''$ is *independent of the direction;* or, taking *one* point and *two* directions, we find the like quotient is *independent of the point.* Hence:

The rectangles of the segments of two intersecting chords are proportional to the squares of the || diameters.

Tangents from any point to a quadric vary as the || diameters.

The areas of sections conjugate to a diameter vary as the rectangles of the segments into which they cut the diameter.

Proof is quite as in Plane Geometry.

42. We have seen that the six terms of second degree are unchanged by a mere change of origin; to find *what functions of the coefficients* are *unchanged by a change of axes*, proceed as in Plane Geometry, thus:

Let the coefficients $k, j, \ldots, i$ change into $k', j', \ldots, i'$, and the sum S of the six terms change into S'; then $S = S'$; also

$$x^2 + 2\,\underline{\omega}xy + y^2 + 2\,\underline{\psi}zx + 2\,\underline{\chi}yz + z^2$$
$$= x'^2 + 2\,\underline{\omega}'x'y' + y'^2 + 2\,\underline{\psi}'z'x' + 2\,\underline{\chi}'y'z' + z'^2,$$

since each is the squared distance D^2 from the common origin O to the same point $P(x, y, z)$ or $P(x', y', z')$. Hence $S + \mu D^2$ is not changed by change of axes; hence the values of μ which make $S + \mu D^2 = 0$ are the *same for all axes*, and specially the values of μ that make $S + \mu D^2$ *resoluble into two factors of first degree* in (x, y, z) are the *same for all axes.* This resolution is possible *when*, and *only when*,

$$\begin{vmatrix} k+\mu & h+\mu\underline{\omega} & g+\mu\underline{\psi} \\ h+\mu\underline{\omega} & j+\mu & f+\mu\underline{\chi} \\ g+\mu\underline{\psi} & f+\mu\underline{\chi} & i+\mu \end{vmatrix} = 0,$$

as was proved in Plane Geometry; for the form of $S+\mu D^2=0$ is the same as that of $F(x, y; x, y)=0$, as is seen on putting $z=1$. The roots of this cubic in μ, called *discriminating cubic* when $\omega=\psi=\chi=90°$, are the *same for all axes;* hence, on making the coefficient of μ^3 1, the **other coefficients** must be **constant** *for all axes.*

They are
$$\{kj-h^2+ki-g^2+ji-f^2$$
$$-2[(hi-gf)\underline{\omega}+(gj-fh)\underline{\psi}+(fk-gh)\underline{\chi}]\}:\boldsymbol{S}^2,$$
$$\{k\cdot\chi|^2+j\cdot\psi|^2+i\cdot\omega|^2$$
$$-2[h(\underline{\omega}-\underline{\chi}\underline{\psi})+g(\underline{\psi}-\underline{\omega}\underline{\chi})+f(\underline{\chi}-\underline{\psi}\underline{\omega})]\}:\boldsymbol{S}^2,$$
$$\begin{vmatrix} k & h & g \\ h & j & f \\ g & f & i \end{vmatrix} : \boldsymbol{S}^2, \quad \text{or} \quad D:\boldsymbol{S}^2.$$

It is to note that the binomials are all co-factors of elements of either $\boldsymbol{S}^2$ or D.

Geometric Interpretation.

43. 1. For *rectangular* axes, $\underline{\omega}$, $\underline{\psi}$, $\underline{\chi}$ vanish, $\omega|$, $\psi|$, $\chi|$ become each $=1$, and $\boldsymbol{S}^2$ becomes 1; hence $k+j+i=$ *constant*. Suppose $S=1$ the central Eq. of a quadric, then k, j, i are the squared reciprocals of the half-diameters; therefore, the *sum of the squared reciprocals of three rectangular diameters of a quadric* is **constant.**

2. If *conjugate diameters* be taken as Cd. axes, h, g, f vanish, and the constants are

$$(kj+ji+ik):\boldsymbol{S}^2,$$
$$(k\cdot\chi|^2+j\cdot\psi|^2+i\cdot\omega|^2):\boldsymbol{S}^2,$$
$$kji:\boldsymbol{S}^2.$$

On dividing the first by the third, we get $\frac{1}{i}+\frac{1}{j}+\frac{1}{k}=constant;$ i.e., *the sum of three squared conjugate* (half-) *diameters of a quadric is* **constant.**

Extracting the second root of the third, and inverting, we get $\boldsymbol{S}:\sqrt{kji}=constant;$ i.e., the *volume of the parallelepiped fixed by three conjugate* (half-) *diameters is* **constant**; hence, its eightfold, the volume of the parallelepiped of three conjugate diameters, or bounded by six tangent planes at the ends of conjugate diameters, is constant.

On dividing the second by the third, there results

$$\frac{\chi|^2}{ij}+\frac{\psi|^2}{ki}+\frac{\omega|^2}{jk}=constant;$$

i.e., the *sum of the squared parallelograms fixed by three conjugate* (half-) *diameters*, taken two by two, is **constant.** The *axes* of the quadric being $2a$, $2b$, $2c$, the values of the above four constants are

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2},\quad a^2+b^2+c^2,\quad abc,\quad \overline{ab}^2+\overline{bc}^2+\overline{ca}^2.$$

The third constant is $abc=a'b'c'\boldsymbol{S}$. Hence $8\pi a'b'c'\boldsymbol{S}$ is *constant;* or, since $\boldsymbol{S}=\omega|\cdot\underline{\zeta}$, $(4\pi a'b'\omega|)(2c'\cdot\underline{\zeta})$ is *constant.* Here the first factor $4\pi a'b'\omega|$ is the area of the central section of the plane XY; and the second, $2c'\cdot\underline{\zeta}$, is the projection of conjugate diameter on the ⊥ to that plane; i.e., the first factor is the base of a *cylinder* touching the quadric along a central section, the bases themselves touching the quadric at the ends of the conjugate diameter; while the second is the height of that cylinder. Such a cylinder may be said to be ‖ *to the diameter.* Hence, *the volume of a circumscribed cylinder* ‖ *to a diameter of a quadric is a* **constant**: $8\pi abc$.

The Special Quadrics.

44. The quadric $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$ has no real points, since the sum of the squares of no three reals can be -1.

The section of the XY-plane is got by putting $z=0$; it is the imaginary ***E*** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1$; all ‖ sections are also imaginary ***E***'s. The like may be said of sections ‖ to the other Cd. planes. Hence the surface may be called *Imaginary* **Ellipsoid**, with axes $2ai$, $2bi$, $2ci$.

The sections of $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ made by the Cd. planes are the real ***E***'s, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, $\frac{x^2}{a^2}+\frac{z^2}{c^2}=1$. All ‖ sections are similar ***E***'s. Hence the surface may be called *Real* **Ellipsoid**, with axes $2a$, $2b$, $2c$. We may suppose $a>b>c$; i.e., $2a$ the *greatest*, $2c$ the *least*, $2b$ the *mean*, axis. For $x>a$, or $y>b$, or $z>c$, the sections become imaginary ***E***'s; hence the surface lies wholly in the parallelepiped whose edges are = and ‖ to the three axes.

45. The plane sections of the ellipsoid are in general *ellipses; are they ever* **circles?** That they are, is made clear geometrically, thus: Pass a plane through the greatest and mean axes; it cuts out the ***E*** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Turn the plane about the mean axis, $2b$ or Y; the section remains an ***E*** of which $2b$ is still the *minor* axis, but the major axis gets *smaller;* when the plane is turned through 90°, the section is the ***E*** $\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, of which $2b$ is the *major* axis. At *some* stage $2b$ must have ceased to be *minor* and become *major;* at *that* stage the axes of the ***E*** were =, the ***E*** was a *circle.* To find the *slope* θ of this *cyclic* plane to the greatest axis, we have the Eq.

$$b^2=c^2:\left(1-\frac{a^2-c^2}{a^2}\cdot \underset{-}{\theta^2}\right),$$

whence $$\theta|=c\sqrt{a^2-b^2}:b\sqrt{b^2-c^2}.$$

Of course there are two *cyclic* central planes, one sloped θ, the other $-\theta$, to the greatest axis. All planes ‖ to these cut the ellipsoid in circles, growing smaller as the cyclic gets farther from the centre, and vanishing in so-called *umbilics*, or, better, *cyclic points*, as the planes become tangents. Clearly there are **four** such points.

When $a = b$, the ellipsoid becomes one of revolution, made by turning the ***E***, whose axes are $2a$, $2c$, about the less axis $2c$; the sections ⊥ to this less axis are all circles clearly, the two series of cyclic sections falling together in them. This ellipsoid is sometimes called an *oblate spheroid*. The earth-surface is nearly such an ellipsoid.

When $b = c$, the ellipsoid, called *prolate spheroid*, is formed by turning an ***E*** about its greater axis $2a$. The two series of cyclic planes fall together ⊥ to the axis $2a$.

46. An important way of looking at the *ellipsoid* is to look at it as a *strained sphere*. Suppose a sphere of radius a to have all its chords ∥ (say) to Y-axis shortened in the ratio $\boldsymbol{b:a}$, and all ∥ (say) to Z shortened in the ratio $\boldsymbol{c:a}$; then, if $P'(x', y', z')$ be any point of the sphere, and OP' be directed by α, β, γ, we shall have

$$x' = \underline{\alpha} a, \quad y' = \underline{\beta} a, \quad z' = \underline{\gamma} a,$$

and if $P(x, y, z)$ be the corresponding point on the surface got by working on the sphere as stated, we shall have

$$x = \underline{\alpha} a, \quad y = \underline{\beta} b, \quad z = \underline{\gamma} c;$$

whence, on squaring and adding, results

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the surface is an *ellipsoid*. We may call α, β, γ the *eccentric* ∡s of P or OP.

Since, in the shortening prescribed, ∥ and = tracts remain ∥ and =, it follows that *conjugate* planes and diameters in the *sphere* remain *conjugate* in the *ellipsoid;* but in the sphere conjugates are ⊥; hence *conjugates* in the *ellipsoid* correspond to ⊥*s* in the *sphere*. Hence, if

$$(\epsilon'_1, \epsilon'_2, \epsilon'_3), \quad (\epsilon''_1, \epsilon''_2, \epsilon''_3), \quad (\epsilon'''_1, \epsilon'''_2, \epsilon'''_3)$$

be eccentric ∡s of three conjugates,

$$\underline{\epsilon}'_1\underline{\epsilon}''_1 + \underline{\epsilon}'_2\underline{\epsilon}''_2 + \underline{\epsilon}'_3\underline{\epsilon}''_3 = 0,$$
$$\underline{\epsilon}''_1\underline{\epsilon}'''_1 + \underline{\epsilon}''_2\underline{\epsilon}'''_2 + \underline{\epsilon}''_3\underline{\epsilon}'''_3 = 0,$$
$$\underline{\epsilon}'''_1\underline{\epsilon}''_1 + \underline{\epsilon}'''_2\underline{\epsilon}''_2 + \underline{\epsilon}'''_3\underline{\epsilon}''_3 = 0.$$

The student may now show that the *sum of the squared projections of three conjugate diameters on any RL. or plane is constant.*

47. The Eq. $\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1$ of the tangent-plane at (x', y', z') becomes in the eccentric form $\frac{x}{a}\underline{\epsilon}'_1 + \frac{y}{b}\underline{\epsilon}'_2 + \frac{z}{c}\underline{\epsilon}'_3 = 0.$ On squaring and adding the Eqs. of three tangent planes at the ends of three conjugate diameters, the locus of the intersection of the three is found to be the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 3.$

48. The *Normal Eq.* of the tangent plane is

$$\underline{\alpha}x + \underline{\beta}y + \underline{\gamma}z = p,$$

where $\underline{\alpha}, \underline{\beta}, \underline{\gamma}$ are position-cosines. Comparing, we see

$$\underline{\alpha} = \frac{px'}{a^2}, \quad \underline{\beta} = \frac{py'}{b^2}, \quad \underline{\gamma} = \frac{pz'}{c^2},$$

and
$$\frac{1}{p^2} = \frac{x'^2}{a^4} + \frac{y'^2}{b^4} + \frac{z'^2}{c^4}.$$

Hence
$$\underline{\alpha}^2a^2 + \underline{\beta}^2b^2 + \underline{\gamma}^2c^2 = p^2\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2}\right) = p^2.$$

Hence the *Normal Eq.* of the tangent planes is

$$\underline{\alpha}x + \underline{\beta}y + \underline{\gamma}z = \sqrt{\underline{\alpha}^2a^2 + \underline{\beta}^2b^2 + \underline{\gamma}^2c^2}.$$

On squaring and adding the Eqs. of three such planes *mutually* ⊥, the locus of the intersection is found to be the *director-sphere* $x^2 + y^2 + z^2 = a^2 + b^2 + c^2$.

49. The quadric $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ is cut by the XY-plane in an ***E*** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, as is seen on putting $z=0$. All ‖ sections are similar ***E***'s, only *larger* the farther from the XY-plane. It is cut by YZ in the ***H*** $\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$. The ‖ sections are similar ***H***'s, *smaller* the further from YZ, till $x=a$, when the section becomes a pair of RLs. $\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$. Thence the sections are secondary ***H***'s, flattening out towards a pair of ‖ RLs. as x nears $\pm\infty$. Like remarks hold for sections ‖ to the XZ-plane. Hence this surface is called an **Hyperboloid** *simple* or *of one sheet.*

By reasoning like that in case of the ellipsoid, it is shown that this hyperboloid is cut in **circles** by a central plane through the *greatest* axis and sloped θ to the *mean* axis, where

$$a^2=-c^2:\left(1-\frac{b^2+c^2}{b^2}\,\theta^2\right),$$

$$\theta|=\pm c\sqrt{a^2-b^2}:a\sqrt{b^2+c^2}.$$

All planes ‖ to these are themselves *cyclic* planes, cutting the surface in ever *larger* circles. Hence, the single hyperboloid has **no** *cyclic points.*

50. The quadric $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c_2}=-1$ is cut by XY in the imaginary ***E*** $\frac{x^2}{a_2}+\frac{y^2}{b^2}=-1$; the ‖ sections remain imaginary till $z=\pm c$; thence the ***E***'s are real and grow ever *larger*, with z nearing $\pm\infty$. The section of YZ is the secondary ***H***', $\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$, and ‖ sections are similar, with ever *larger* axes. Like remarks hold for sections ‖ to XZ. Hence, this surface is called an **Hyperboloid** *double* or *of two sheets.*

The student can readily convince himself that the *cyclic* planes of the *simple* hyperboloid are also *cyclic* planes of the *double*; the circular sections shrink into *four cyclic points* as they retire from the centre.

These two hyperboloids have clearly a common *asymptotic cone* $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$, which has common *cyclic planes* with them.

51. The clearest notion of these three surfaces is got thus: Turn an equiaxial H, its conjugate H', and their common asymptotes around the conjugate axis; the H will trace out an equiaxial simple hyperboloid of revolution, the H' an equiaxial double hyperboloid of revolution, the asymptotes the common equiaxial asymptotic cone of revolution. Now change (say shorten) all chords $\perp$ to ZX in the ratio $b:a$; all chords $\perp$ to XY in the ratio $c:a$; the resulting surfaces will be the surfaces in question.

The circle $\frac{x^2}{a^2}+\frac{y^2}{a^2}=1$ traced by the vertex of the revolving H is called *circle of the gorge*; the corresponding E $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is called *ellipse of the gorge*.

52. Passing now to **non-centric quadrics**, we see that the **first** $\frac{x^2}{a}+\frac{y^2}{b}=4z$ is cut by XY in the point $(0, 0, 0)$, the origin, while all $\parallel$ sections are E's: real for $z>0$, imaginary for $z<0$. The section of YX is the P $y^2=4bz$, while that of ZX is the P $x^2=4az$. The surface may be thought made by a variable E moving always $\parallel$ to XY, with its vertices on these two P's. The surface is called **Elliptic Paraboloid**; $4a$ and $4b$ are its *parameters*; O is its *vertex*. Suppose the P $y^2=4az$ to turn around its axis, the Z-axis; the surface generated will be the paraboloid of revolution $\frac{x^2}{a^2}+\frac{y^2}{a^2}=4z$. Now suppose all y's, or all chords $\perp$ to XZ, shortened in the ratio $b:a$; the surface got so is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=4z$.

As to **cyclic planes**, we may reason thus: Turn a plane ∥ to XZ about the major axis of its elliptic section; the minor axis grows, and the ***E*** tends to a ***P***, as the plane turns through 90°; at *some* ∡ the *minor* must have become *equal* to the *major* axis, the ***E*** must have passed over into a *circle*. The slope of the cyclic planes to XY is readily seen to be θ when $\theta = \pm\sqrt{b:a}$, $a > b$.

The student will readily see there are **two** *cyclic points*.

53. The **second** *non-centric* $\frac{x^2}{a} - \frac{y^2}{b} = 4z$ is likewise tangent to XY at O, as is seen on writing out the Eq. of plane tangent at (0, 0, 0): $\frac{x \cdot 0}{a} - \frac{y \cdot 0}{b} = 2(z+0)$, or $z = 0$, which is the XY-plane. But XY cuts the surface along the pair of RLs. $\frac{x^2}{a} = \frac{y^2}{b}$. All ∥ sections are ***H***'s: primary for $z > 0$, secondary for $z < 0$. The sections of YZ and ZX are the ***P***'s $y^2 = -4bz$, $x^2 = 4az$. The surface may be thought made by an ***H*** moving, always ∥ to XY, with its vertices on one of these two ***P***'s. In all positions the asymptotes of the ***H*** are ∥ to the pair $\frac{x^2}{a} = \frac{y^2}{b}$. As the ***H*** nears the XY-plane, it passes over into this pair of RLs., then into its conjugate, while its vertices pass over from one ***P*** on to the other. To two *counter*-values, $+z$, $-z$, correspond two conjugate ***H***'s. The surface is named **Hyperbolic Paraboloid**; $4a$ and $4b$ are its *parameters*, O is its *vertex*, Z is its *axis;* it is *saddle-like* in shape. (See Figs. at end.)

54. It is easy to see geometrically that the cyclic planes thus far determined are *all* the *real* ones. For the *diametral* plane of the chords of the circles must be ⊥ to them; hence it must be one of the three *chief* planes; hence the diameter of the *central circle* must be one of the *axes*. This can only be the *mean* one, $2b$, in case of the ellipsoid; for any circle of radius a

resp. c lies wholly *without* resp. *within* the ellipsoid. Similar reasoning holds for the other surfaces. But there are other *imaginary cyclic planes*, as may thus be shown analytically:

Be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ an ellipsoid, and $\frac{x^2}{r^2}+\frac{y^2}{r^2}+\frac{z^2}{r^2}=1$ a concentric sphere; then

$$\left(\frac{1}{a^2}-\frac{1}{r^2}\right)x^2+\left(\frac{1}{b^2}-\frac{1}{r^2}\right)y^2+\left(\frac{1}{c^2}-\frac{1}{r^2}\right)z^2=0$$

is a cone through O, being homogeneous of second degree, and through the intersection of sphere and ellipsoid, being satisfied whenever their Eqs. are. *When, and only when*, this cone breaks up into **two planes**, the intersection of ellipsoid and sphere is a *plane* curve; i.e., is a **circle**. This is the case only when the Eq. is resoluble into two linear factors; and this is the case only when the determinant Δ vanishes, or when one coefficient (one element in the diagonal of Δ, the others being 0) vanishes; and this is so only when $r^2=a^2$, or b^2, or c^2. For $r^2=a^2$ or $r^2=c^2$, the factors, i.e., the planes, are *imaginary;* for $r^2=b^2$ they are *real*. The student can easily apply the reasoning to the other surfaces.

55. We have seen that *two* RLs. lie on the *hyperbolic paraboloid:* the intersection of that surface and the XY-plane. But the general proposition holds:

All surfaces of second degree are **ruled**. On each lies an ∞ of RLs. This is clear at once on referring to the condition that a RL. lie on a surface (Art. 17): on combining the Eqs. of RL. and surface, the resultant Eq. in a single Cd. must *vanish identically*. This Eq., being of second degree, vanishes thus when its **three** coefficients each reduce to 0; and these **three** Eqs. of condition can be satisfied by the **four** parameters of a RL. in an ∞ of ways.

56. Let us apply this argument to the *simple hyperboloid:*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

Be $\quad y = tz + v, \quad x = sz + u$

the RL. On substitution results

$$\left(\frac{s^2}{a^2}+\frac{t^2}{b^2}-\frac{1}{c^2}\right)z^2+2\left(\frac{su}{a^2}+\frac{tv}{b^2}\right)z+\frac{u^2}{a^2}+\frac{v^2}{b^2}=1.$$

This vanishes identically, is satisfied for every z, when

$$\frac{s^2}{a^2}+\frac{t^2}{b^2}-\frac{1}{c^2}=0, \quad \frac{su}{a^2}+\frac{tv}{b^2}=0, \quad \frac{u^2}{a^2}+\frac{v^2}{b^2}=1.$$

Hence result readily the *real* values

$$s = \pm\, av : bc, \quad t = \mp\, bu : ac.$$

The third Eq. of condition says that every such RL. meets the *ellipse of the gorge*, as was to be foreseen. The *double* sign shows that through every point of this ellipse go **two** RLs.; there lies on the surface a *double* system of RLs. Two RLs., one of each system, are

$$x = \frac{av}{bc}z + u, \quad y = -\frac{bu}{ac}z + v,$$

and $$x = -\frac{av'}{bc}z + u', \quad y = \frac{bu'}{ac}z + v'.$$

The condition that these two RLs. *meet* is (Art. 9)

$$u - u' : v - v' = \frac{av}{bc}+\frac{av'}{bc} : -\left(\frac{bu}{ac}+\frac{bu'}{ac}\right),$$

a condition always fulfilled, since

$$\frac{u^2}{a^2}+\frac{v^2}{b^2}=\frac{u'^2}{a^2}+\frac{v'^2}{b^2}=1.$$

Hence **every RL. of each system meets every RL. of the other.**

Changing the signs of $\frac{av'}{bc}$ and $\frac{bu'}{ac}$, we see that the condition is fulfilled only when $(u-u')^2b^2 = -a^2(v-v')^2$; i.e., **never**. Hence **no RL. of either system meets any RL. of the same system.** Hence through every point of the surface there pass *two*, and *only two*, RLs. on it. The *plane of these RLs.* is clearly the *plane tangent* at their intersection. For it can meet the surface only on these RLs., which form its conic of intersection; hence all RLs. in it through the intersection of the pair meet the surface only at that point. As the point of tangence *glides along* either of the RLs., the tangent plane *turns about* the RL., cutting the surface.

The RLs. of either system are called *elements* (or generators) of the surface. Since no two *elements* meet, the surface is not *torse* or *developable*, but *skew* or a *scroll*. This is easy to see, thus: Be 1, 2, 3, 4, ... consecutive elements. *If* 1 and 2 meet, 2 and 3 meet, 3 and 4 meet, etc., then we may turn the strip between 1 and 2, which is plane, *infinitesimally*, about 2 till it falls into the plane of the strip 2 3; then turn the sum of the strips 1 2 and 2 3 about 3 into the plane of the strip 3 4, and so on. Thus, and thus only, could the surface be turned off, unwrapped, into a plane surface. Now, since 1 and 2, 2 and 3, etc., do *not* meet, this can *not* be done. A more rigorous proof would not be in place here.

57. To find the RLs. on an *ellipsoid*, in the values of s and t, put c^2 for $-c^2$, or ic for c; the values then fall out *imaginary*: there are *no real* RLs. on the *ellipsoid*.

On putting ai, bi for a, b, the values of s and t again fall out *imaginary*: *no real RLs.* lie on the *double hyperboloid*.

58. Proceeding with the *hyperbolic paraboloid* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=4z$ exactly as with the simple hyperboloid, we find for s and t the *real* values: $s=a:u$, $t=\pm\sqrt{ab}:u$. Hence there lies on it a double system of real RLs. Every RL. of either system cuts every RL. of the other. No RL. of either system cuts a RL. of the same system. Through every point of the surface pass a pair of RLs., fixing the tangent plane at the point, which plane cuts the surface. The surface is not developable.

Eliminating z, the student will find the XY-projection of an element to be

$$y=\pm\sqrt{\frac{b}{a}}(x-2u);$$

whence projections of all elements, on XY, are ∥ to one of the pair of RLs. $y = \pm\sqrt{b:a}\cdot x$, in which XY cuts the surface; i.e., all elements are ∥ to one of the planes fixed by Z and this pair. These planes contain the asymptotes of the generating hyperbola. Hence *all elements of a hyperbolic paraboloid are* ∥ *to an asymptotic plane.* (See Figs. at end.)

59. The *foci* of the **chief** *sections* of a quadric are called **foci** of the *quadric*. In an ellipsoid with half-axes a, b, c, the section (ab) is an ***E*** with two foci F, F' on the axis $2a$, distant $\sqrt{a^2-b^2}$ from the centre; the section (bc) is an ***E*** with two foci G, G' on the axis $2b$, distant $\sqrt{b^2-c^2}$ from the centre; the section (ac) is an ***E*** with two foci H, H' on the axis $2a$, distant $\sqrt{a^2-c^2}$ from the centre. Thus, on the greatest axis lie four foci, on the mean axis lie two, on the least lie *none.*

If a quadric with half-axes a', b', c' be **confocal** with this base-ellipsoid, the relations hold:

$$a'^2 - b'^2 = a^2 - b^2, \quad b'^2 - c'^2 = b^2 - c^2, \quad a'^2 - c'^2 = a^2 - c^2;$$

whence

$$a'^2 - a^2 = b'^2 - b^2 = c'^2 - c^2 = (\text{say})\ \lambda;$$

i.e., $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1,$ and $\dfrac{x^2}{a^2+\lambda}+\dfrac{y^2}{b^2+\lambda}+\dfrac{z^2}{c^2+\lambda}=1,$

are *confocal for all values of* λ.

To trace the system: for $\lambda = +\infty$ the surface is a sphere with radius ∞; as λ sinks toward $-c^2$, the surface (an ellipsoid) shrinks, and for $\lambda = -c^2$ flattens to the **inner** *doubly-laid* surface of the so-called *focal* ***E*** in the section (ab), whose half-axes are $\sqrt{a^2-c^2}$, $\sqrt{b^2-c^2}$, its foci F, F', and its vertices H, H', G, G'; as λ sinks from $-c^2$ towards $-b^2$, the *outer doubly-laid* surface (thought as a simple hyperboloid) spreads out into a simple hyperboloid, which, as λ nears $-b^2$, flattens into the so-called *focal* ***H*** in the section (ac) with foci H, H' and vertices F, F'; as λ sinks from $-b^2$, the surface becomes a double hyperboloid, which, as λ nears $-a^2$, flattens down to the section (bc); as λ sinks from $-a^2$ towards $-\infty$, the surface becomes and remains an *imaginary ellipsoid.*

60. For any triplet (x', y', z') the Eq. of the confocal yields three values of λ: $\lambda_1, \lambda_2, \lambda_3$; by reasoning quite like that in Plane Geometry (Art. 148), it is shown that these roots lie between $+\infty$ and $-c^2$, $-c^2$ and $-b^2$, $-b^2$ and $-a^2$, resp.; i.e., *through every point of space pass three, and only three, confocals:* an ellipsoid, a simple hyperboloid, and a double hyperboloid.

The three λ's are called *elliptic Cds.* of the point (x', y', z'). Substituting them in the Eq. of the confocal, in turn, and solving the three Eqs. as to x', y', z', we get

$$x' = \sqrt{\{\overline{a^2+\lambda_1}\cdot\overline{a^2+\lambda_2}\cdot\overline{a^2+\lambda_3}\}} : \sqrt{\{\overline{a^2-b^2}\cdot\overline{a^2-c^2}\}},$$

and two like expressions for y', z' got by permuting a, b, c.

If we divide this x' by $a^2+\lambda_1$, we get the coefficient of x in the Eq. of the plane tangent at (x', y', z') to the first confocal; dividing it by $a^2+\lambda_2$, we get the corresponding coefficient in the Eq. of the plane tangent to the second confocal; the product of these two coefficients is $(a^2+\lambda_3):(a^2-b^2)(a^2-c^2)$. The products of the coefficients of y and of z in the two Eqs. are got by simply permuting a, b, c. The sum of these three products is 0. This means, by Art 16, that the *two planes are* $\perp$. Like holds, of course, for the second and third confocals, and for the third and first. Hence **three confocals through a point are mutually $\perp$.**

Cubature of the Quadric.

61. *The part of a surface intercepted between two* ∥ *planes* is called a **zone.** The *space* bounded by the *planes* and the zone we may call a **segment** of the *surface* (meaning a *segment* of the *space* fixed by the *surface*).

Suppose an equiaxial H, its conj. H', and their common asymptotes turned about the conjugate axis. There will be generated by H, *a simple hyperboloid* of revolution; by H', a *double hyperboloid* of revolution; by the asymptotes, a *cone* of revolution: the Eqs. are

$$x^2+y^2-z^2=a^2, \quad x^2+y^2-z^2=-a^2, \quad x^2+y^2-z^2=0.$$

Sections of the surfaces ⊥ to Z are circles, and their areas, they being distant z from XY, are

$$\pi(z^2+a^2), \qquad \pi(z^2-a^2), \qquad \pi z^2.$$

Hence it is clear that the circle of the *cone* is the *arithmetic mean* between the circles of the *hyperboloids;* it differs from each of these by a ring whose area is πa^2, the area of the *circle of the gorge;* these differ from each other by *double* this area, by $2\pi a^2$. It is to note that the circle of the double hyperboloid is *imaginary*, and so does not really come into consideration, for $z < a$.

Accordingly, to find the volume of any *hyperboloidal segment*, it suffices to find the volume of the corresponding *cone-segment* and then *add* resp. *subtract* the volume of the corresponding *ring-segment* in case of the single resp. double hyperboloid. The cone-segment is itself the difference of two cones whose altitudes are (say) z_1, z_2, and bases πz_1^2, πz_2^2; hence the volume is $\frac{\pi}{3}(z_2^3-z_1^3)$; the constant area of a section of the ring-space is πa^2, and the altitude is z_2-z_1; hence the volume is $\pi a^2(z_2-z_1)$.

If the **H** be *not* equiaxial, but have axes $2a$, $2c$, then to any altitude z will correspond in the cone a circle of radius not z but $\frac{a}{c}z$, the surfaces then being

$$x^2+y^2-\frac{a^2}{c^2}\cdot z^2=a^2, \qquad x^2+y^2-\frac{a^2}{c^2}\cdot z^2=-a^2,$$

$$x^2+y^2-\frac{a^2}{c^2}\cdot z^2=0.$$

Hence it is enough to change z into $az:c$. From these last surfaces the most general, viz.,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1,$$

$$\frac{x^2}{a^2}+\frac{y}{b^2}-\frac{z^2}{c^2}=0,$$

are got by shortening every y in the ratio $b:a$; hence it suffices to multiply the preceding results by this ratio.

62. The ellipsoid (a, b, c) is got from the sphere (a, a, a) by shortening every y in the ratio $b:a$, and every z in the ratio $c:a$; hence the whole volume of the ellipsoid is got from that of the sphere by shrinking it in the ratio $bc:a^2$; and the same ratio holds between volumes of corresponding parts of ellipsoid and sphere. The volume of the sphere is $\frac{4}{3}\pi a^3$, hence that of ellipsoid is $\frac{4}{3}\pi abc$.

On three axes, $2a$, $2b$, $2c$, construct an *ellipsoid* and the two *hyperboloids;* also construct a *cylinder* tangent to the *simple* hyperboloid along the *ellipse of the gorge*, its bases tangent to the *double* hyperboloid at the latter's vertices. Let us compare the volumes E, C, K, H of the ellipsoid, cylinder, cone-segment, hyperboloid-segment, the bases of the two latter being the bases of the cylinder. The volume E is $\frac{4}{3}\pi abc$; C is $2c\cdot\pi ab$ or $2\pi abc$; K is $\frac{1}{3}$ of C, or is $\frac{2}{3}\pi abc$; H is K plus $2c\cdot\pi ab$, or H is $\frac{8}{3}\pi abc$; hence

$$K:E:C:H=1:2:3:4.$$

63. *To find the volume V of a segment of the* **elliptic paraboloid** $\frac{x^2}{a}+\frac{y^2}{b}=4z$, first take the vertex for one base and the section of the plane $z=z$ for the other; cut this cap-shaped segment into n thin slices by planes ‖ to the base; let the altitude of each slice be $\frac{z}{n}$; it will have two bases, each an E, a *greater* and a *less;* the volume of each slice will be *less* than the altitude by the *greater* base and *greater* than the altitude by the *less* base; hence the *whole volume* will be less than the common altitude $\frac{z}{n}$ by the sum of the *greater* bases and *greater* than that altitude by the sum of the *smaller* bases; or, common factors set out,

$$V < \frac{z^2}{n^2} \cdot 4\pi\sqrt{ab}\{1+2+3+\cdots+n\},$$

$$V > \frac{z^2}{n^2} \cdot 4\pi\sqrt{ab}\{0+1+2+\cdots+\overline{n-1}\},$$

or
$$V < \frac{4\pi\sqrt{ab}\cdot z\cdot z}{2}\left\{1+\frac{1}{n}\right\},$$

$$V > \frac{4\pi\sqrt{ab}\cdot z\cdot z}{2}\left\{1-\frac{1}{n}\right\}.$$

As n nears ∞, $\frac{1}{n}$ nears 0, and there results

$$\boldsymbol{V = 2\pi\sqrt{ab}.z.z.}$$

Now $4\pi\sqrt{ab}\cdot z$ is the base of the segment, z its altitude; hence $4\pi\sqrt{ab}\cdot z\cdot z$ is the volume of the circumscribing cylinder; hence the *volume of a cap-segment of an elliptic paraboloid is* **half** *that of the circumscribed cylinder.*

The volume of any segment is the difference of two cap-segments.

64. To find the volume v of a segment of an *hyperbolic paraboloid* $\frac{x^2}{a^2} - \frac{y^2}{b} = 4z$, suppose it bounded by the surface, the XY-plane, the YZ-plane, and a plane $x = x$, $\parallel$ to YZ. The section of this last plane with the surface is a parabola; the chord (in XY-plane) of the segment of this $\boldsymbol{P}$ is $\boldsymbol{2x\sqrt{b:a}}$, the altitude of the segment is $\boldsymbol{x^2:4a}$; hence the area is

$$\frac{\sqrt{ab}\,.\,x^3}{3\,.\,a^2}.$$

Cut the solid segment into n thin slices; then, reasoning exactly as before, we get

$$V < \frac{\sqrt{ab}\cdot x^4}{3\cdot a^2 n^4}\{1^3+2^3+3^3+\cdots+n^3\},$$

$$V > \frac{\sqrt{ab}\cdot x^4}{3\,a^2\cdot n^4}\{0^3 + 1^3 + 2^3 + \cdots + \overline{n-1}^3\}.$$

Now $$1^3 + 2^3 + \cdots + n^3 = \frac{\overline{n(n+1)}^2}{4},$$

as we know from Algebra; hence

$$V < \frac{\sqrt{ab}\cdot x^4}{12\,a^2}\left(1+\frac{1}{n}\right)^2,$$

$$V > \frac{\sqrt{ab}\cdot x^4}{12\,a^2}\left(1-\frac{1}{n}\right)^2.$$

As n nears ∞, $\frac{1}{n}$ nears 0, and there results

$$V = \sqrt{ab}\cdot x^4 : 12\,a^2.$$

Now $\sqrt{ab}\cdot x^3 : 3\,a^2$ is the base of the segment, x its altitude; hence $\sqrt{ab}\cdot x^4 : 3\,a^2$ is the volume of the circumscribed cylinder; hence the *volume of such a segment of an hyperbolic paraboloid is one-fourth that of the circumscribed cylinder.*

The student may confirm the results as to the ellipsoid and the hyperboloids by this method of slices.

The segments thus far treated have been *right*, i.e., $\perp$ to an axis of the surface; but like reasoning applies to oblique segments, on observing that the intercept between the bases on the conjugate diameter is not the altitude of the segment but a multiple of it.

Varieties of Quadrics.

65. If a quadric be given by its Eq. in the general form, it is of course possible to determine what kind of a quadric it is by reducing its Eq. to the simplest form; but this is tedious. It is possible to establish certain simple tests, however, by some such reasoning as this:

By Art. 30 the surface is *centric* or *non-centric*, according as $D \gtrless 0$ or $D = 0$. If $D \gtrless 0$, the centric Eq. is

$$kx^2 + 2\,hxy + jy^2 + 2\,gzx + 2\,fyz + iz^2 + \frac{\Delta}{D} = 0.$$

If now a RL. through the centre, $y = tz$, $x = uz$, meet the surface in finity for all values of t and u, the surface is closed or *ellipsoidal;* otherwise, it is *hyperboloidal.* The student can show that the first holds when both $D>0$ and C or $kj - h^2 > 0$; the surface is then a **real** *ellipsoid*, an **imaginary** *cone* with one real point (the centre), or an **imaginary** *ellipsoid*, according as $\Delta<0$, $\Delta=0$, or $\Delta>0$ (k being taken + always).

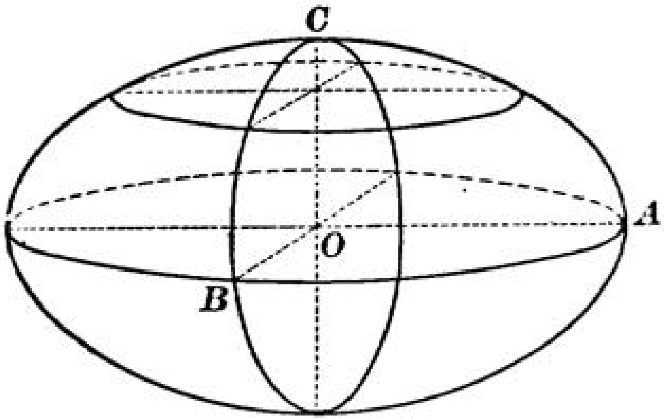
If D and C be not both >0, the surface is hyperboloidal. We now test whether the RLs. on it be real or imaginary by the former method; the result is, the surface is a **simple** *hyperboloid*, an **elliptic** *cone*, or a **double** *hyperboloid*, according as

$$\Delta<0, \quad \Delta=0, \quad \Delta<0.$$

In case $D=0$, the surface is *non-centric* or *paraboloidal.* Putting $z=0$, we find the section of the XY-plane is an ***E*** or an ***H***, i.e., the surface is an **elliptic** or an **hyperbolic** *paraboloid*, according as $C>0$ or $C<0$. If $C=0$, the section is a ***P***, and this test fails. In that case, test with $ik - g^2$ in the same way. If *both* $kj - h^2$ and $ik - g^2$ vanish, then must also $ji - f^2$ vanish; all sections are ***P***'s, and the surface is a *parabolic* **cylinder.**

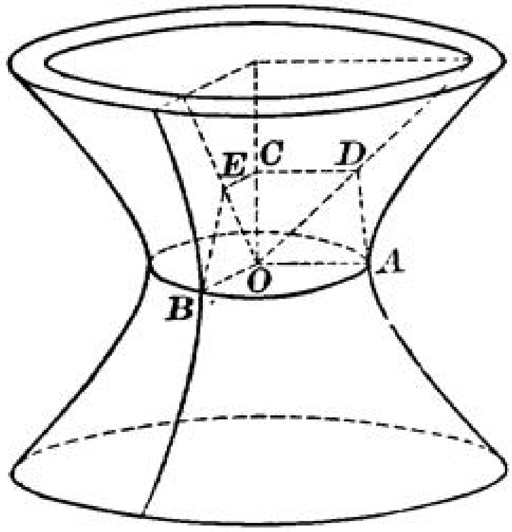
Lastly, in case one or more of the numerators L, M, N of the Cds. of the centre ($L:D$, $M:D$, $N:D$), as well as the common denominator D, vanish, the centre becomes indeterminate, the surface has an ∞ of centres. The surface is then a **cylinder.** In case the three Eqs. of planes $Q_x=0$, $Q_y=0$, $Q_z=0$ which determine the centre reduces to two only, their line of intersection is the *line of centres*, every point on it is a centre of the surface. The cylinder is *elliptic*, *hyperbolic*, or breaks up into a *pair of planes*, according as one of its plane sections is an ***E***, an ***H***, or a *pair of* intersecting *RLs.* In case the three Eqs. reduce to one, each represents the *plane of centres*, every point on it is a centre of the surface. The surface itself consists of two ∥ planes, midway between which lies the ∥ *plane of centres.* In case the Eq. of the surface is a *perfect square*, the surface consists of *two planes fallen together* in the *plane of centres.*

Fig. 1



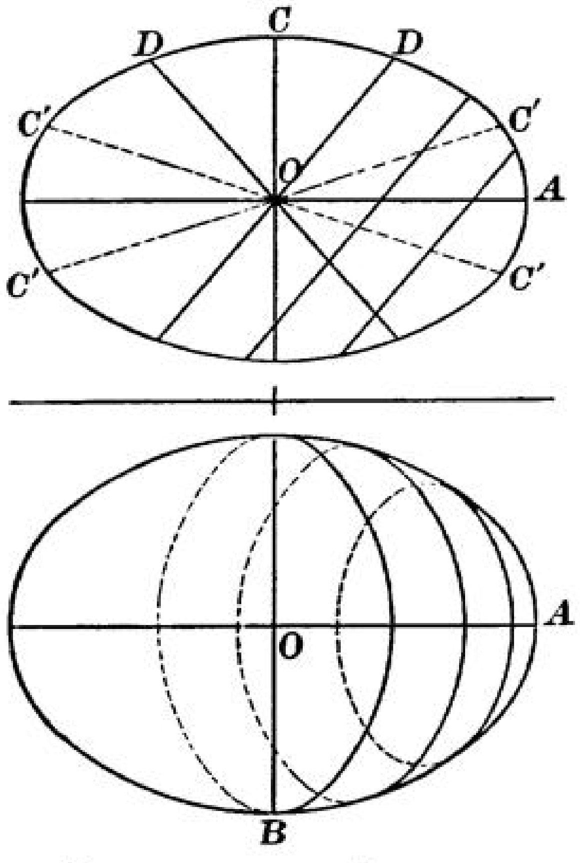
ELLIPSOID.
$OA=a,\ OB=b,\ OC=c.$

Fig. 3



SIMPLE HYPERBOLOID.
AB is Ellipse of the Gorge.
EOD is the Asymptotic Cone.

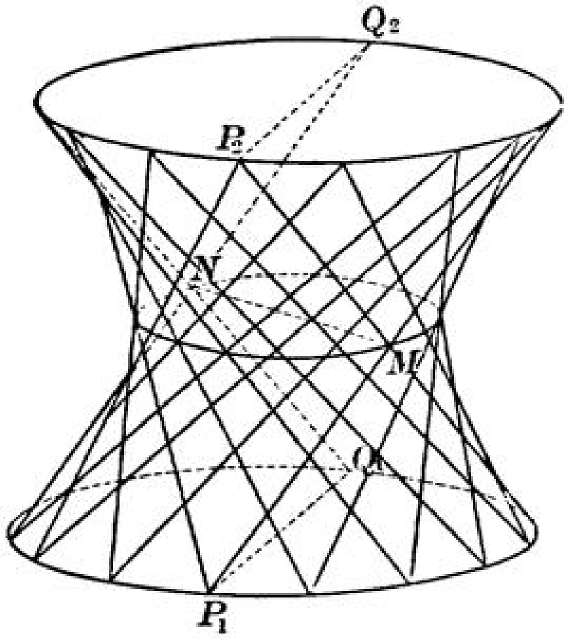
Fig. 2



PROJECTIONS OF ELLIPSOID ON *XZ* AND *XY*.

OD gives direction of cyclic sections.
C′s are cyclic points.
C′C′ is locus of centres of cyclic sections.

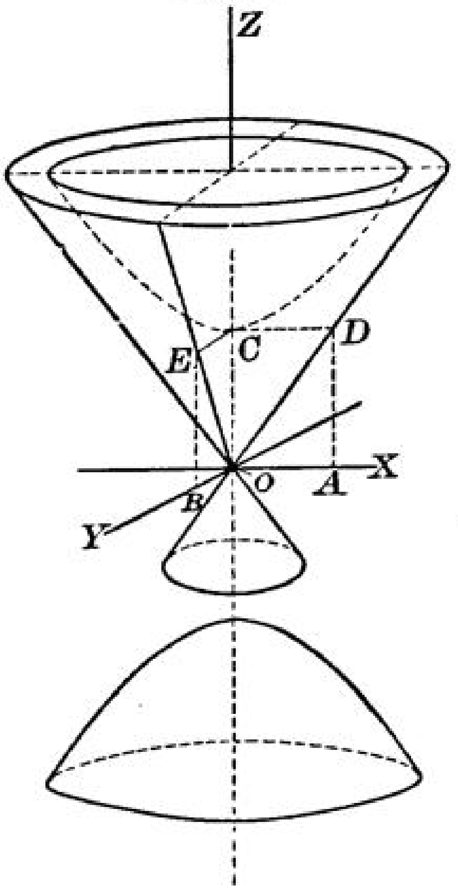
Fig. 4



RLs. ON THE S. H.
MP_1 and NQ_1 are of 1st system,
MP_2 and NQ_2 are of 2d system.

NOTE. For Figures on pp. 280 and 281 thanks are due Fort and Schloemilch's *Analytische Geometrie.*

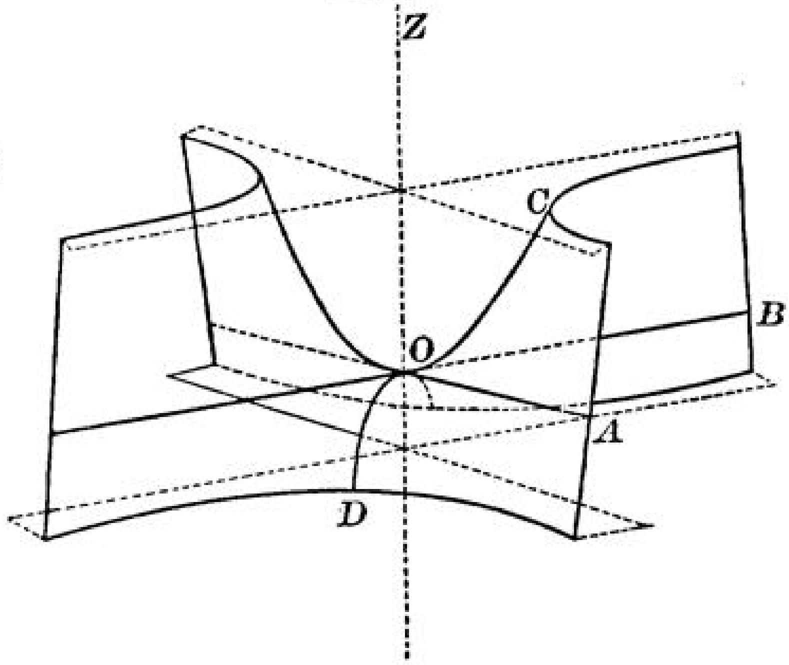
Fig. 5



DOUBLE HYPERBOLOID.

EOD is the Asymptotic Cone.

Fig. 7

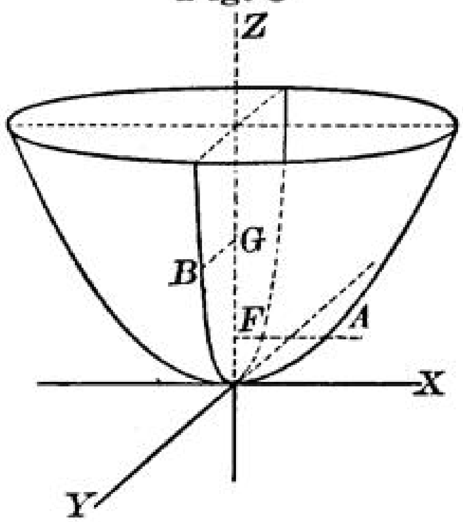


HYPERBOLIC PARABOLOID.

OC and OD are Parabolas.

OA and OB are Asymptotic directions for the Hyperbolas.

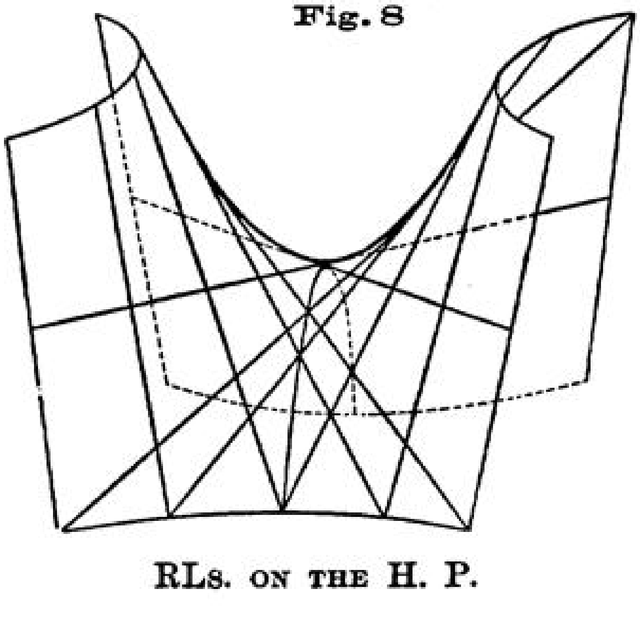
Fig. 6



ELLIPTIC PARABOLOID.

$FA = 2a$ and $GB = 2b$ are half-parameters.

Fig. 8



RLs. ON THE H. P.

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Elements of the Integral Calculus.

By W. E. BYERLY, Professor of Mathematics in Harvard University. 8vo. 204 pages. Mailing Price, $2.15; Introduction, $2.00.

This volume is a sequel to the author's treatise on the Differential Calculus (see page 134), and, like that, is written as a text-book. The last chapter, however, — a Key to the Solution of Differential Equations, — may prove of service to working mathematicians.

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Elements of the Differential and Integral Calculus.

With Examples and Applications. By J. M. TAYLOR, Professor of Mathematics in Madison University. 8vo. Cloth. 249 pp. Mailing price, $1.95; Introduction price, $1.80.

The aim of this treatise is to present simply and concisely the fundamental problems of the Calculus, their solution, and more common applications. Its axiomatic datum is that the change of a variable, when not uniform, may be conceived as becoming uniform at any value of the variable.

It employs the conception of rates, which affords finite differentials, and also the simplest and most natural view of the problem of the Differential Calculus. This problem of *finding the relative rates of change of related variables* is afterwards reduced to that of finding the limit of the ratio of their simultaneous increments; and, in a final chapter, the latter problem is solved by the principles of infinitesimals.

Many theorems are proved both by the method of rates and that of limits, and thus each is made to throw light upon the other. The chapter on differentiation is followed by one on direct integration and its more important applications. Throughout the work there are numerous practical problems in Geometry and Mechanics, which serve to exhibit the power and use of the science, and to excite and keep alive the interest of the student.

Judging from the author's experience in teaching the subject, it is believed that this elementary treatise so sets forth and illustrates the highly practical nature of the Calculus, as to awaken a lively interest in many readers to whom a more abstract method of treatment would be distasteful.

Oren Root, Jr., *Prof. of Math., Hamilton Coll., N.Y.:* In reading the manuscript I was impressed by the clearness of definition and demonstration, the pertinence of illustration, and the happy union of exclusion and condensation. It seems to me most admirably suited for use in college classes. I prove my regard by adopting this as our text-book on the calculus.

C. M. Charrappin, S.J., *St. Louis Univ.:* I have given the book a thorough examination, and I am satisfied that it is the best work on the subject I have seen. I mean the best work for what it was intended,—a text-book. I would like very much to introduce it in the University. (*Jan.* 12, 1885.)